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CONVEX REGIONS OF STATIONARY SPACETIMES AND RANDERS SPACES. APPLICATIONS TO LENSING AND ASYMPTOTIC FLATNESS

ERASMO CAPONIO, ANNA VALERIA GERMINARIO, AND MIGUEL SÁNCHEZ

ABSTRACT. By using *stationary-to-Randers correspondence* (SRC, see [20]), a characterization of light and time-convexity of the boundary of a region of a standard stationary $(n + 1)$ -spacetime is obtained, in terms of the convexity of the boundary of a domain in a Finsler n or $(n + 1)$ -space of Randers type. The latter convexity is analysed in depth and, as a consequence, the causal simplicity and the existence of causal geodesics confined in the region and connecting a point to a stationary line are characterized. Applications to asymptotically flat spacetimes include the light-convexity of hypersurfaces $S^{n-1}(r) \times \mathbb{R}$, where $S^{n-1}(r)$ is a sphere of large radius in a spacelike section of an end, as well as the characterization of their time-convexity with natural physical interpretations. The *lens effect* of both light rays and freely falling massive particles with a finite lifetime, (i.e. the multiplicity of such connecting curves) is characterized in terms of the focalization of the geodesics in the underlying Randers manifolds.

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1. INTRODUCTION

Assume that a stellar object emitted radiation (light rays or massive particles, the latter possibly with a finite lifetime) in the past, and we can assume that the spacetime is stationary on a region which includes that source and ourselves. When can we ensure that we will receive such a radiation and, in this case, when will it be focalized towards us, obtaining so multiple images of the source? The key ingredients will be the existence of a *light or time convex hypersurface* around the region, plus the possibility of lensing effects due to either curvature or topology.

Convexity is a basic property of subsets of Euclidean space which admits several extensions to semi-Riemannian manifolds. For a domain D of a complete Riemannian manifold, its convexity will mean geodesic connectedness by means of *minimizing* geodesics in D . This is equivalent to the *local* and *infinitesimal* convexity of its boundary ∂D , when ∂D is regular enough (see complete details below). For a Lorentzian manifold (M, g) , these notions for ∂D admit a straightforward analog, even extensible to causality types (time, space and lightlike convexities). Nevertheless, the notion of convexity for D is not so clear, as just geodesic connectedness makes sense in general, but local *extremal* properties appear only for geodesics of causal type. However, for (conformally) stationary spacetimes, the recent progress in its causal structure [20] (which is related with elements of Finsler Geometry, [18, 20]), plus the developments on convexity in the Finslerian setting [3], allow to carry out a detailed explanation of convexity in this Lorentzian setting, to be studied here.

Recall that a spacetime is called *stationary* if it admits a timelike Killing vector field Y . Locally, each stationary spacetime (L, g_L) admits a *standard form* (namely, $L = S \times \mathbb{R}$, $g_L = g_0 + 2\omega_0 dt - \beta dt^2$, for some *lapse* β and *shift* ω_0 , see below) and, when this form can be obtained globally, the spacetime is called *standard stationary*. This structure will be assumed here, and it is not too restrictive, as the completeness of Y plus the property of being distinguishing for L (a causality condition less restrictive than strong causality) ensure it [41]. Standard stationarity becomes natural in the framework of asymptotically flat spacetimes, including black holes, as the “no hair” results postulate that these will stabilize in some member of the Kerr family. Moreover, as our techniques will be conformally invariant in some cases, the corresponding results will be extensible to spacetimes such as the classical FLRW ones.

Even though connections with Finsler Geometry were pointed out long time ago [43, 44], results about causality of standard stationary spacetimes obtained by using an accurate relation with such a geometry, have been obtained only recently (see [18, 20]). This relation is based on the fact that the projections on S of lightlike geodesics in (L, g_L) are pregeodesics for a suitable Finsler metric on S of Randers type, called *Fermat metric* in [18]. Such a simple property leads to a kaleidoscope of relations between the Causality of standard stationary spacetimes and the geometry of Randers spaces, or *stationary-to-Randers correspondence (SRC)*, carefully developed in [20].

In this paper, our aim is to use SRC in order to describe the convexity of a stationary region $D \times \mathbb{R}$ of a standard stationary spacetime $L = S \times \mathbb{R}$. The light-convexity and time-convexity of $D \times \mathbb{R}$ are characterized in terms of the convexity w.r.t. the geodesics of a Randers metric on, resp., D and the product $\mathbb{R}_u \times D$, where $\mathbb{R}_u \equiv \mathbb{R}$ (the subindex u will be used throughout the paper to recall the

natural coordinate on the factor $(\mathbb{R}, du^2)$, not to be confused with the global time coordinate t associated to other copy of $\mathbb{R}$). Moreover, these elements are also characterized in terms of the existence of a causal geodesic with fixed length $l \geq 0$ (which is intended not to be bigger than the mean lifetime of the travelling particle) connecting a point and an integral line of the Killing field Y and *minimizing the arrival time t* . These results lie in the general framework of relativistic lensing ([59]) and complement the classical Fermat principle ([42, 58]) which assures that, in the case that a time minimizing connecting lightlike curve exists, then it must be a lightlike geodesic –but it does not assure existence. We emphasize that our results are not merely sufficient conditions to ensure that the connecting causal geodesic will exist. On the contrary, the Fermat metric provides the geometric framework to fully characterize their existence in both cases, lightlike geodesics and timelike geodesics with a prescribed length.

The paper is organized as follows. In Section 2, after a summary on the notion of convexity for a Finsler manifold (including the recent progress in [3]), the convexity of the domain D for a Randers metric $R = \sqrt{h} + \omega$ is characterized (Proposition 2.5) and discussed (Examples 2.7, 2.8). As a consequence, the convexity of large balls in asymptotically flat Randers spaces is shown (Proposition 2.9). For this result, only the decay of $d\omega$ (rather than ω) becomes relevant (formula (14)).

In Section 3, the convexity of domains $D \times \mathbb{R}$ in a standard stationary spacetime $L = S \times \mathbb{R}$ is characterized. For light-convexity, the infinitesimal convexity of $\partial(D \times \mathbb{R})$ becomes equivalent to the infinitesimal convexity of ∂D with respect to the Fermat metric F (Theorem 3.4). Then, SRC and the results in [3] yield easily the equivalence with the local notion of light-convexity (Corollary 3.6). For time-convexity, a further insight is obtained by using the fact that timelike geodesics can be obtained as projections of lightlike geodesics of a product manifold $\mathbb{R}_u \times L$ of one dimension more. The equivalence between the infinitesimal time-convexity of the boundary $\partial(D \times \mathbb{R})$ and the infinitesimal convexity of $\mathbb{R}_u \times \partial D$ for a suitable Randers metric F_β is detailed (Theorem 3.9). Moreover, the infinitesimal convexity of the latter hypersurface is characterized (Proposition 3.11).

These results are applied to asymptotically flat stationary spacetimes in Section 4. Concretely, a notion of asymptotic flatness (Definition 4.1) specially adapted to this setting, is introduced and discussed along Subsection 4.1. In the next subsection, the light-convexity of the hypersurfaces $S^{n-1}(r) \times \mathbb{R}$ for r sufficiently large is proven (Corollary 4.4), and the hypotheses under which time-convexity holds (or is violated) are provided (Corollary 4.7). Remarkably, time-convexity will not hold for large spheres $S^{n-1}(r)$ under general physical assumptions (Proposition 4.9), in contrast with the lightlike case. In Subsection 4.3, the paradigmatic case of Kerr spacetime is analysed specifically. The non time-convexity of $S^{n-1}(r) \times \mathbb{R}$, for all large enough r , is interpreted (Remark 4.11), and the hypersurfaces close to the stationary limit one are also taken into account (Corollary 4.12).

In the first subsection of Section 5, a full characterization of the problem of connecting a point (p, t_p) and a stationary line $l_q = \{(q, t) : t \in \mathbb{R}\}$ by means of a (first-arriving) future-pointing lightlike geodesic contained in a stationary domain $D \times \mathbb{R}$ is obtained (Theorem 5.3). This is characterized alternatively in terms of: (a) Geometric/Variational interpretations of the boundary: light-convexity of $\partial D \times \mathbb{R}$, (b) Finsler geometry: convexity of D with respect to the associated Fermat metric, and (c) Causal structure: causal simplicity of the domain $D \times \mathbb{R}$. When

D is not contractible, infinitely many connecting lightlike geodesics (with diverging arrival times) appear. This can be interpreted as a *topological lens effect* (while the gravitational lensing depends strictly on the curvature of the Fermat metric). As emphasized in Remark 5.4, these conclusions and the usage of SRC here, complete the circle of results and techniques in papers on boundaries such as [32, 35], where variational methods are applied to the study of Lorentzian geodesics¹. Finally, in Remark 5.6, further physical applicability of the results is pointed out.

In Subsection 5.2, previous results are extended to timelike geodesics. From the technical viewpoint, the following difficulty is worth pointing out. Our main result (Theorem 5.7) is proved by using an auxiliary product spacetime $\mathbb{R}_u \times L$. Nevertheless, our hypotheses are posed naturally on the original stationary domain D , rather than on the auxiliary elements in the product spacetime. For the connection between the hypotheses on these two spacetimes (see Lemma 5.10), a small improvement on the results of convexity for Finslerian metrics is carried out (Remark 5.2). By using this method, Theorem 5.7 assures the existence of a connecting future-pointing timelike geodesic with *a priori* fixed Lorentzian length and minimizing the arrival time t at the stationary curve l_q . Removing the minimizing property, when D is not contractible one obtains also the multiplicity of such connecting timelike geodesics. That is, any freely falling massive particle, starting at some event p , will be able to reach l_q and, if D is not contractible, arriving after unbounded values of time t , *even if the lifetime of the particle is arbitrarily small*.

Due to the technical subtleties of our approach, in Subsection 5.3 a revision of the available causal, topological and variational tools for this kind of problems, is carried out. We stress how stationary-to-Randers Correspondence fits with the other techniques to provide a complete solution of causal geodesic connectedness in the stationary setting, and point out further related problems.

Finally, in the last section the conclusions are summarized.

2. CONVEXITY OF DOMAINS OF RANDERS MANIFOLDS

2.1. Finsler metrics. Let us recall some notions about Finsler manifolds. A Finsler structure on a smooth (connected) manifold M of dimension n is a function $F: TM \rightarrow [0, +\infty)$ which is continuous on TM , smooth on $TM \setminus 0$, vanishing only on the zero section, fiberwise positively homogeneous of degree one (i.e. $F(\lambda y) = \lambda F(y)$, for all $y \in TM$ and $\lambda > 0$), and which has fiberwise strongly convex square, that is, the matrix

$$g_y = \left[\frac{1}{2} \frac{\partial^2 (F^2)}{\partial y^i \partial y^j} (y) \right] \quad (1)$$

is positive definite for any $y \in TM \setminus 0$. Observe that $y \in TM \setminus 0 \mapsto g_y$ is a symmetric section of the tensor product of the pulled back cotangent bundle $\pi^* T^* M$ over $TM \setminus 0$ with itself. Henceforth, besides to the quite standard notation $A(y)$, we will also use – to get more compact formulas – an index that indicates the dependence on $y \in TM \setminus 0$ for sections A of $\pi^* T^* M$, its dual $\pi^* T^* M$ or their tensor product (for

¹Recall that these techniques were initiated in [11] with the introduction of time and light convexity in the static case. This case becomes quite simpler, as it is related to Riemannian instead of properly Finslerian metrics, see [6]. The techniques also apply to periodic trajectories and other Lorentzian variational problems on convex domains (see the subtleties in [7] and references therein).

example, g_y above or $(H_\phi)_y$ for the Hessian, with respect to the Chern connection, of a function ϕ on M).

The minimal requirement about the regularity of F that we need is that the fundamental tensor g is $C_{\text{loc}}^{1,1}$ in $TM \setminus 0$.

By homogeneity, $F(y) = g_y(y, y)$, for all $y \in TM$, thus the fundamental tensor g gives the shape of the unit sphere (indicatrix), $F(y) = 1$, $y \in T_x M$, at each point $x \in M$. Hence, its positive definiteness yields the strict convexity of the closed ball $\bar{B}_x = \{y \in T_x M : F(y) \leq 1\}$, that is any line segment joining two points contained in $\bar{B}_x$ is contained in B_x , except, at most, its endpoints (see, for example, [1, Exercise 2.1.6]).

The length of a piecewise smooth curve $\gamma : [a, b] \rightarrow M$ with respect to the Finsler metric F is defined by

$$\ell_F(\gamma) = \int_a^b F(\dot{\gamma}) \, ds$$

hence the Finsler distance between two arbitrary points $p, q \in M$ is given by

$$d(p, q) = \inf_{\gamma \in \mathcal{P}(p, q; M)} \ell_F(\gamma),$$

where $\mathcal{P}(p, q; M)$ is the set of all piecewise smooth curves $\gamma : [a, b] \rightarrow M$ with $\gamma(a) = p$ and $\gamma(b) = q$. The distance function is non-negative and satisfies the triangle inequality, but in general it is not symmetric since F is only *positively* homogeneous in y , that is, d is a *generalized distance* (see [28] for an exhaustive study). As a consequence, the *reverse* Finsler metric of F is defined as $\tilde{F}(y) = F(-y)$. So, for any point $p \in M$ and for all $r > 0$, we can define two different balls centered at p and having radius r : the *forward ball* $B^+(p, r) = \{q \in M \mid d(p, q) < r\}$ and the *backward* one $B^-(p, r) = \{q \in M \mid d(q, p) < r\}$. Analogously, it makes sense to introduce two different types of Cauchy sequences and completeness: a sequence $(x_n)_n \subset M$ is a *forward* (resp. *backward*) *Cauchy sequence* if for all $\varepsilon > 0$ there exists an index $\nu \in \mathbb{N}$ such that for all $m \geq n \geq \nu$, it is $d(x_n, x_m) < \varepsilon$ (resp. $d(x_m, x_n) < \varepsilon$); consistently a Finsler manifold is *forward complete* (resp. *backward complete*) if every forward (resp. backward) Cauchy sequence converges². In general, the backward elements for F are forward for $\tilde{F}$, and we will refer just to forward elements. It is well known that the topology generated by the forward balls coincides with the underlying manifold topology. Moreover, an adapted version of the Hopf-Rinow theorem holds (cf. [1, Theorem 6.6.1]) stating, in particular, the equivalence between forward completeness and compactness of closed and forward bounded subsets (i.e., those included in some forward ball) of M .

2.2. Convexity. We say that a Finsler manifold (M, F) is *(geodesically) convex* if each pair of points $(p, q) \in M \times M$ can be connected by a (non-necessarily unique) minimizing geodesic, i.e. a geodesic with length $d(p, q)$, starting at p and ending at q . Recall that convexity for F is equivalent to convexity for $\tilde{F}$. Any of the assumptions of the Hopf-Rinow theorem (namely, either forward or backward completeness) imply convexity. However, after [20], it becomes clear that the forward or the backward completeness of the generalized metric d can be substituted in several

²It is worth pointing out that a second natural notion of forward and backward Cauchy sequence can be given, see [28, Section 3.2.2]. This notion is *not* equivalent to that stated above, but it yields equivalent (forward and backward) Cauchy completions and, thus, equivalent notions of completeness.

classical results (as convexity, Bonnet-Myers or Synge theorems) by the assumption of the compactness of the closed balls with respect to the symmetrized distance d_s associated to d , namely

$$d_s(p, q) = \frac{1}{2} (d(p, q) + d(q, p)), \quad \forall p, q \in M.$$

More precisely, let B_s denote the balls with respect to d_s . If the closed balls $\bar{B}_s(x, r)$ are compact for all $x \in M$ and $r > 0$ (or equivalently the subsets $\bar{B}^+(x, r_1) \cap \bar{B}^-(y, r_2)$ are compact for any $x, y \in M, r_1, r_2 > 0$), then (M, F) is convex, [20, Theorem 5.2]. It is worth to stress that the Hopf-Rinow theorem does not hold in general for the metric d_s . For instance, Example 2.3 in [20] exhibits a non compact, d_s -bounded Randers space whose symmetrized distance d_s is complete.

From a variational viewpoint, geodesics parametrized with constant speed (i.e. $s \mapsto F(\gamma(s), \dot{\gamma}(s)) = \text{const.}$) and connecting two fixed points p and q on (M, F) , are the critical points of the energy functional

$$J(\gamma) = \frac{1}{2} \int_a^b F^2(\dot{\gamma}) \, ds$$

defined on the manifold of the H^1 curves γ on M , parametrized on the interval $[a, b] \subset \mathbb{R}$ and such that $\gamma(a) = p, \gamma(b) = q$ (see, for example, [18, Proposition 2.1]).

The convexity of a domain (i.e., an open connected subset) $D \subset M$, regarded as a Finsler manifold in its own right, can be related to the *infinitesimal convexity* of its boundary ∂D , at least when the closure $\bar{D}$ is a manifold with boundary and ∂D is at least twice continuously differentiable, i.e. when ∂D is (locally and then globally) the inverse image of a regular value of some C^r function with $r \geq 2$. Infinitesimal convexity means that for each $x \in \partial D$ there exists a neighborhood $U \subset M$ of x such that for one (and then for all) C^2 function $\phi : U \rightarrow \mathbb{R}$ such that

$$\begin{cases} \phi^{-1}(0) = U \cap \partial D \\ \phi > 0 & \text{on } U \cap D \\ d\phi(x) \neq 0 & \text{for every } x \in U \cap \partial D \end{cases} \quad (2)$$

one has

$$(H_\phi)_y(y, y) \leq 0 \quad \text{for every } y \in T_x \partial D \setminus \{0\}, \quad (3)$$

where H_ϕ is the Hessian of ϕ with respect to the Chern connection ∇ of (M, F) , i.e. $H_\phi = \nabla(d\phi)$ (see [1, Section 2.4]) More precisely, in natural coordinates on $TM \setminus \{0\}$, $(H_\phi)_y(u, v)$ is given by (the Einstein summation convention is used in the remainder)

$$((H_\phi)_y)_{ij} u^i v^j = \frac{\partial^2 \phi}{\partial x^i \partial x^j} u^i v^j - \frac{\partial \phi}{\partial x^k} \Gamma_{ij}^k(y) u^i v^j, \quad (4)$$

where $\Gamma_{ij}^k = \Gamma_{ij}^k(y)$ are the components of the Chern connection given by

$$\Gamma_{ij}^k = \frac{g^{ks}}{2} \left(\frac{\delta g_{si}}{\delta x^j} - \frac{\delta g_{ij}}{\delta x^s} + \frac{\delta g_{js}}{\delta x^i} \right).$$

Here, $g_{ks} = g_{ks}(y)$ and $g^{ks} = g^{ks}(y)$ are, respectively, the components of g_y (see (1)) and of its inverse at $y \in TM \setminus 0$, while $\frac{\delta}{\delta x^i}$ are vector fields on TM defined as

$$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N_i^j(y) \frac{\partial}{\partial y^j},$$

where $N_i^j = N_i^j(y)$ are the components of the so-called *non-linear connection on* $TM \setminus 0$ (see [1, §2.3]). As the equation of a geodesic $\gamma = \gamma(s)$, parametrized with constant speed, i.e. $F(\dot{\gamma}) = \text{const.}$, is given by

$$\frac{d^2 \gamma^i}{ds^2} + \Gamma_{jk}^i(\dot{\gamma}) \dot{\gamma}^j \dot{\gamma}^k = 0,$$

it is immediate to see that

$$(\phi \circ \gamma)''(s) = (H_\phi)_{\dot{\gamma}(s)}(\dot{\gamma}(s), \dot{\gamma}(s)). \quad (5)$$

Remark 2.1. Eq. (5) holds also for a Riemannian or a Lorentzian metric g , namely if $\gamma = \gamma(s)$ is a geodesic of g then $(\phi \circ \gamma)''(s) = H_\phi(\dot{\gamma}, \dot{\gamma})$, where, in this case, the Hessian of ϕ is $H_\phi = \nabla(d\phi)$ and ∇ is the Levi-Civita connection of g .

Remark 2.2. This notion of infinitesimal convexity for a hypersurface is a natural extension to the Finslerian setting of the analogous one in a Riemannian manifold. Let us summarize the relation between this notion and the ones of *local* and *strong* convexity for ∂D . Recall that, on one hand, an embedded hypersurface N is *locally convex* when for each $x \in N$ a small enough neighborhood U of x exists such that all the geodesics in U issuing from x and tangent to N lie in the closure of one of the two connected parts of $U \setminus N$, called the *local exterior*. Recall that, as the hypersurface ∂D in (2) is the boundary of a domain, a standard argument based on partitions of the unity shows that all the locally defined functions can be taken so that they match in a global one (see, for example, [61]) and the global exterior (namely, $M \setminus D$) is well defined. On the other hand, when (3) is satisfied with the strict inequality, we will say that ∂D is *strongly convex* at $x \in \partial D$.

Trivially, strong convexity at a point implies both, local and infinitesimal convexity on a neighborhood of that point, and it is also clear that the local convexity at a point implies the infinitesimal one at the same point (cf. e.g. [64, Prop. 14.2.1 and Th. 14.2.3]). The non-triviality of the last converse when the inequality (3) is not strict (for that, one needs also to assume that the inequality holds on a neighborhood of the point, otherwise the implication is not true), was stressed by Bishop [13] (see also the review [61]), who proved this equivalence in the Riemannian case for a C^4 metric. The proof of the equivalence in the general Finsler case was obtained recently in [3, Corollary 1.2], where, the degree of differentiability was also lowered to $C_{\text{loc}}^{1,1}$ (i.e. C^1 on $TM \setminus \{0\}$ with locally Lipschitz differential) for the fundamental tensor g and $C_{\text{loc}}^{2,1}$ for the function ϕ .

Remark 2.3. We point out that a refinement of the proof in [3] on the equivalence between infinitesimal convexity and local one allows to optimize the degree of differentiability of the hypersurface ∂D to C^2 (see [17]).

Finally, recall also that, following [3], when the subsets $\bar{B}_s(x, r) \cap \bar{D}$ (or just the intrinsic d_s -closed balls of $\bar{D}$) are compact for all $x \in D, r > 0$, the above equivalent notions of convexity for ∂D become also equivalent to the convexity of D , seen as a Finsler manifold in itself with the metric induced by F (see, Theorem 5.1 and Remark 5.2 below).

Remark 2.4. Using the terminology in [40, Defn. 2.3], whenever (3) holds for all $y \in T(U \cap \partial D)$, we can say that D is *convex* (w.r.t. the Finsler metric) *near* x . Clearly, the function $-\phi$ is a *local defining function* for the domain D in the sense of [40, Definition 2.1] and, as explained in Remark 2.2, one can assume that the

function ϕ is defined on a neighborhood of the boundary ∂D . Remarkably, in [40] the authors show that for an open subset D of $\mathbb{R}^n$, having a C^2 , infinitesimally convex boundary (or, equivalently, such that D is convex, cf. also Theorem 5.1) there exist various locally defining C^2 functions which are also convex (i.e. their Hessians are positive semi-definite for all $p \in U$ and all $y \in \mathbb{R}^n$). In particular, for any bounded D such that ∂D is infinitesimally convex, the signed distance to the boundary gives a defining function which is convex on a tubular neighborhood of ∂D , see [37, Lemma 14.17]. To extend such a result for Riemannian manifolds, further hypotheses would be required, as non-positive curvature for the ambient manifold. In fact, for a round sphere S^2 , an open hemisphere gives a domain whose boundary is totally geodesic (and hence, infinitesimally convex) but no convex defining function can exist in a neighborhood of its boundary, as convex functions must be constant on closed geodesics.

2.3. Randers spaces. In this paper, we deal with the convexity of a domain in a Randers space. Given a Riemannian manifold (S, h) and a one-form ω on S such that, for any $x \in S$, $\|\omega\|_x < 1$, where $\|\omega\|_x = \sup_{y \in T_x S \setminus \{0\}} |\omega(y)| / \sqrt{h(y, y)}$, a Randers metric R and its reversed one $\tilde{R}$ on S are defined by setting

$$R(y) = \sqrt{h(y, y)} + \omega(y) \quad \text{and} \quad \tilde{R}(y) = \sqrt{h(y, y)} - \omega(y), \quad y \in TS \quad (6)$$

Condition $\|\omega\|_x < 1$ is necessary and sufficient for R and $\tilde{R}$ to be positive and it implies that they have fiberwise strongly convex square (see [1, §11.1] for details).

The condition (3) at a point of ∂D in a Randers space (S, R) can be written in terms of the Hessian H^h of ϕ with respect to the Levi-Civita connection of the Riemannian metric h plus another term involving $d\omega$.

In what follows, ∇^h will denote the gradient symbol with respect to the metric h as well as and the Levi-Civita connection of h and $\widehat{d\omega}$ the $(1, 1)$ -tensor field h -metrically associated to $d\omega$, i.e. for every $(x, y) \in TM$, $d\omega_x(\cdot, y) = h_x(\cdot, \widehat{d\omega}(y))$.

Proposition 2.5. *Let D be a domain of class C^2 of a Randers manifold (S, R) , $x \in \partial D$, ϕ be a function defined on a neighborhood of x satisfying conditions (2). Then, the following propositions are equivalent:*

- (i) $(\partial D; R)$ is infinitesimally convex at x ;
- (ii) $(\partial D; \tilde{R})$ is infinitesimally convex at x ;
- (iii) for all $y \in T_x \partial D$

$$H_\phi^h(y, y) + \sqrt{h(y, y)} d\omega(y, \nabla^h \phi) \leq 0; \quad (7)$$

- (iv) for all $y \in T_x \partial D$

$$H_\phi^h(y, y) - \sqrt{h(y, y)} d\omega(y, \nabla^h \phi) \leq 0; \quad (8)$$

- (v) for all $y \in T_x \partial D$

$$H_\phi^h(y, y) + \sqrt{h(y, y)} |d\omega(y, \nabla^h \phi)| \leq 0. \quad (9)$$

Proof. It is enough to show that inequality (7) is equivalent to the inequality (3). Indeed the other equivalences will follow simply by the definition (6) of R and $\tilde{R}$ and observing that (8) is the evaluation of (7) in $-y$. If γ is any smooth curve on S , we can compute the second derivative of $\phi \circ \gamma$ by using the Levi-Civita connection

of h . As $(\phi \circ \gamma)' = h(\nabla^h \phi, \dot{\gamma})$ and $H_\phi^h(\dot{\gamma}(s), \dot{\gamma}(s)) = h(\nabla_{\dot{\gamma}(s)}^h(\nabla^h \phi(\gamma(s))), \dot{\gamma}(s))$ (see e.g. [55, Ch. 3, Lemma 49]), we obtain

$$\begin{aligned} (\phi \circ \gamma)''(s) &= h(\nabla_{\dot{\gamma}(s)}^h(\nabla^h \phi(\gamma(s))), \dot{\gamma}(s)) + h(\nabla^h \phi(\gamma(s)), \nabla_{\dot{\gamma}(s)}^h \dot{\gamma}(s)) \\ &= H_\phi^h(\dot{\gamma}(s), \dot{\gamma}(s)) + h(\nabla^h \phi(\gamma(s)), \nabla_{\dot{\gamma}(s)}^h \dot{\gamma}(s)). \end{aligned} \quad (10)$$

A geodesic $\gamma = \gamma(s)$ of (S, R) (parametrized with constant Randers speed), satisfies in particular the pregeodesic equation (see for example the computations in [19] above its Eq. (6)):

$$\nabla_{\dot{\gamma}}^h \dot{\gamma} = \sqrt{h(\dot{\gamma}, \dot{\gamma})} \widehat{d\omega}(\dot{\gamma}) + \frac{1}{2} \frac{d}{ds} (\log(h(\dot{\gamma}, \dot{\gamma}))) \dot{\gamma}. \quad (11)$$

Now, for any $x \in \partial D$ and $y \in T_x \partial D \setminus \{0\}$ consider the geodesic γ such that $\gamma(0) = x$ and $\dot{\gamma}(0) = y$. As $h_x(\nabla^h \phi(x), y) = 0$, substituting (11) in (10) and recalling (5), we obtain

$$(H_\phi)_y(y, y) = H_\phi^h(y, y) + \sqrt{h(y, y)} h(\nabla^h \phi, \widehat{d\omega}(y)),$$

i.e., the expression in the left-hand side of (7). $\square$

Clearly, Proposition 2.5 can be extended to strong convexity.

By (7), the Hessians of ϕ for R and h will agree on the vectors tangent to ∂D if and only if $d\omega(\nabla^h \phi, \cdot)$ vanishes there. However, from (9) the following holds:

Corollary 2.6. *If $(\partial D; R)$ is infinitesimally convex then also $(\partial D; h)$ is infinitesimally convex.*

The following example shows that the converse is not true.

Example 2.7. Let $S = \mathbb{R}^2$ be endowed with a Randers metric as in (6), being h the usual Euclidean metric and ω defined as $\omega_x(y) = f(x^2)y^1$ for any $x = (x^1, x^2)$, $y = (y^1, y^2) \in \mathbb{R}^2$, where $f : \mathbb{R} \rightarrow \mathbb{R}$ is a smooth function such that $|f| < 1$. For any $r_0 > 0$, take as a domain the open ball

$$D_{r_0} = \{(x^1, x^2) \in \mathbb{R}^2 \mid (x^1)^2 + (x^2)^2 < r_0^2\}$$

whose boundary is defined by $\phi(x^1, x^2) = r_0^2 - (x^1)^2 - (x^2)^2$. Obviously, $\nabla^h \phi(x) = (-2x^1, -2x^2)$, $H_\phi^h(y, y) = -2(y^1)^2 - 2(y^2)^2$, and ∂D_{r_0} is convex with respect to h for any $r_0 > 0$. Nevertheless, it is easy to find cases where $(\partial D_{r_0}; R)$ is not convex. For example, it is enough to assume $r_0|f'(r_0)| > 1$ (in addition to $|f| < 1$). Indeed, $d\omega(y, z) = f'(x^2)y^1z^2 - f'(x^2)y^2z^1$, and the convexity condition (9) can be written as

$$-(y^1)^2 - (y^2)^2 + \sqrt{(y^1)^2 + (y^2)^2} |f'(x^2)| |x^2 y^1 - x^1 y^2| \leq 0 \quad (12)$$

for any $x \in \partial D_{r_0}$, $y \in T_x \partial D_{r_0}$. But (12) is not fulfilled for $x = (0, r_0)$, $y = (1, 0)$.

Next, the previous example is modified in order to show that the convexity of $(\partial D; R)$ does not imply the convexity of ∂D with respect to the Riemannian metric $h_0 = h - \omega^2$. This question becomes natural because, on one hand, $\sqrt{h_0} + \omega^2 + \omega$ defines always a Randers metric (with no restriction on ω), and, on the other, such an h_0 becomes the Riemannian metric on the slices of a standard stationary spacetime (see Remark 3.5(2) below).

Example 2.8. Redefine, in Example 2.7, the 1-form as $\omega_x(y) = f(x^1)y^1$. Since $d\omega = 0$, $(\partial D_{r_0}; R)$ is convex from Proposition 2.5. Let $h_0 = h - \omega^2$, i.e.,

$$h_{0(x^1, x^2)}((y^1, y^2), (y^1, y^2)) = (1 - f(x^1)^2)(y^1)^2 + (y^2)^2.$$

To check that $(\partial D_{r_0}; h_0)$ is not convex for simple choices of f , recall that a curve $\gamma(s) = (x^1(s), x^2(s))$, $s \in I$, is a geodesic for h_0 iff

$$\begin{cases} \ddot{x}^1 = \frac{f(x^1)f'(x^1)}{1-f(x^1)^2}(\dot{x}^1)^2 \\ \ddot{x}^2 = 0. \end{cases}$$

So the h_0 -Hessian of ϕ is

$$H_\phi^{h_0}((y^1, y^2), (y^1, y^2)) = -2 \left(1 + x^1 \frac{f(x^1)f'(x^1)}{1-f(x^1)^2} \right) (y^1)^2 - 2(y^2)^2. \quad (13)$$

Notice that, as (y^1, y^2) is assumed to be tangent to ∂D_{r_0} at (x^1, x^2) , necessarily $y^2 = -x^1 y^1 / x^2$ whenever $x^2 \neq 0$. So, the right part of (13) reads:

$$-2 \left(1 + x^1 \frac{f(x^1)f'(x^1)}{1-f(x^1)^2} + \frac{(x^1)^2}{r_0^2 - (x^1)^2} \right) (y^1)^2.$$

Thus, for each $x^1 \neq \pm r_0, 0$ and any choice of $f(x^1)$, we can choose $f'(x^1) \neq 0$ so that (13) becomes positive for $y^1 \neq 0$.

2.4. Convexity in asymptotically flat Randers manifolds. The notion of asymptotic flatness is specially relevant for Riemannian manifolds, and it allows to ensure that large balls are convex. Next we explore the analogous issues for a Randers manifold.

Consider a Riemannian manifold (S, h) endowed with a one-form ω , with $\|\omega\|_x < 1$, for each $x \in S$. Assume that there is a compact set $K \subset S$ such that $S \setminus K$ is a disjoint union of ends, $E^{(k)}$, $k = 1, \dots, m$, such that each end is diffeomorphic to $\mathbb{R}^n \setminus \{0\}$ and in each end there exist a coordinate chart $x = (x^1, \dots, x^n)$ and positive constants p and q such that h and ω satisfy

$$\begin{aligned} h_{ij} &= \delta_{ij} + O(1/|x|^p), \\ \partial_k h_{ij} &= O(1/|x|^{p+1}), \\ \Omega_{ij} &:= \partial_i \omega_j - \partial_j \omega_i = O(1/|x|^{q+1}), \end{aligned} \quad (14)$$

as $|x| \rightarrow +\infty$, where $|\cdot|$ denotes the natural norm in each coordinate chart, i.e., $|x|^2 = \sum_{i=1}^n (x^i)^2$. Obviously, we can take always $p = q$ in (14) because if the inequalities held for, say, $0 < q < p$ then all of them will hold for q too. However, we leave both exponents to stress the asymptotic behaviours of h and ω . In fact, the above growth assumptions on h , plus bounds on its second derivatives (including the scalar curvature), are commonly used to define *asymptotic flatness* in a purely Riemannian setting (see e.g. [63] or [15]). We will not require bounds neither for the second derivatives of h nor for its scalar curvature, as convexity involves only pregeodesics (i.e. the connection rather than the curvature, see also Remark 3.3). Consistently, we require only bounds for the differential of ω and not for ω itself, because if an exact form is added to ω the pregeodesics remains unchanged. For these reasons, when (14) are fulfilled on each end we say that the Randers space is *geodesically asymptotically flat* (see also Remark 4.3 below). The charts where (14) holds will be called *asymptotic coordinates* and, if one of such charts exists, then so will be any chart obtained from the first one by a transformation with Jacobian matrix equal to $r_{ij} + O(1/|x|^{p_0})$ and derivatives $O(1/|x|^{1+p_0})$ outside a compact set, where r_{ij} is an orthogonal matrix and $p_0 > 0$ any constant.

Let us focus now on large spheres $S^{n-1}(r_0)$, defined in the asymptotic coordinates of each end $E^{(k)}$, as $|x|^2 = r_0^2$, $r_0 > 0$ large enough. Let $\phi_{r_0}^{(k)}(x) = r_0^2 - |x|^2$ and

$$D_{r_0}^{(k)} = \{x \in E^{(k)} \mid \phi_{r_0}^{(k)}(x) > 0\}. \quad (15)$$

Let D_{r_0} be the domain of S equal to $\bigcup_k D_{r_0}^{(k)} \cup K$ and $\phi(\equiv \phi_{r_0})$ be an extension of all the $\phi_{r_0}^{(k)}$'s to S . Let us see that the boundary of D_{r_0} is convex if r_0 is big enough. Let $\gamma = \gamma(s)$ be a geodesic of the Randers metric defined by h and ω . If γ is parametrized with constant Randers speed $\sqrt{h(\dot{\gamma}, \dot{\gamma})} + \omega(\dot{\gamma}) = \text{const.}$ then, in local coordinates, γ satisfies the following equation (compare with (11))

$$\ddot{\gamma}^l = -\Gamma_{ij}^l \dot{\gamma}^i \dot{\gamma}^j + \sqrt{h(\dot{\gamma}, \dot{\gamma})} h^{lm} (\partial_m \omega_j - \partial_j \omega_m) \dot{\gamma}^j + \frac{1}{2} \frac{d}{ds} (\log(h(\dot{\gamma}, \dot{\gamma}))) \dot{\gamma}^l, \quad (16)$$

where Γ_{ij}^l are the components of Levi-Civita connection of h and h^{kl} is the inverse of h_{kl} . Let $x \in \partial D_{r_0}^{(k)}$ and $y \in T_x \partial D_{r_0}^{(k)}$, arguing as in the proof of Proposition 2.5 and using (16) we get,

$$(H_\phi)_y(y, y) = -2|y|^2 + 2x^k h_{kl} \left(\Gamma_{ij}^l y^i y^j - \sqrt{h(y, y)} h^{lm} (\partial_m \omega_j - \partial_j \omega_m) y^j \right). \quad (17)$$

As $h_{ij} = \delta_{ij} + O(1/|x|^p)$, its inverse h^{ij} is of the type

$$h^{ij} = \delta^{ij} + O(1/|x|^p) \quad (18)$$

and then, recalling the second condition in (14), $\Gamma_{ij}^l = O(1/|x|^{p+1})$; thus from (17) we get

$$(H_\phi)_y(y, y) \leq -2|y|^2 + 2C \left(\frac{1}{|x|^{p+1}} + \frac{1}{|x|^{q+1}} \right) |x| |y|^2 \quad (19)$$

which is negative, if $|x| = r_0$ is large enough. Summing up, we get the following

Proposition 2.9. *In any geodesically asymptotically flat Randers manifold (in the sense specified in formula (14)) ∂D_{r_0} is strongly convex, for any sufficiently large enough r_0 .*

3. STATIONARY SPACETIMES AND CAUSALLY CONVEX BOUNDARIES

3.1. Background and SRC. Randers spaces are deeply related to the causal structure of stationary Lorentzian manifolds. We start recalling some basic definitions and notations (see [8, 39, 52, 55] for further information).

A *Lorentzian manifold* is a pair (L, g_L) where L is a smooth (connected) manifold and g_L a metric on L of index one, with signature $(+, \dots, +, -)$. A non-zero tangent vector $v \in T_z L$, $z \in L$, is said *timelike* (respectively *lightlike*; *spacelike*) when $g_L(v, v) < 0$, (respectively $g_L(v, v) = 0$; $g_L(v, v) > 0$) and *causal* if it is timelike or lightlike. A *spacetime* is a Lorentzian manifold (L, g_L) endowed with a time-orientation. The latter is determined by some timelike vector field Y , so that a causal vector $v \in T_z L$ is said *future-pointing* (resp. *past-pointing*) if $g_L(v, Y) < 0$ (resp. $g_L(v, Y) > 0$). A piecewise smooth curve $z : [a, b] \rightarrow L$ is said timelike, lightlike or spacelike if so is $\dot{z}(s)$ at any $s \in [a, b]$ where it exists. In particular, non-constant geodesics z are classified according to the sign of $g_L(\dot{z}, \dot{z})$.

A spacetime (L, g_L) is said *stationary* if it admits a timelike Killing vector field Y . In this case, one such a Y that points to the future will be chosen and called the *stationary vector field*. When Y is complete and L satisfies a mild causality condition (to be distinguishing, which lies between causality and strong causality),

then L will be *standard stationary* (see [41, Proposition 3.1]). More precisely, L splits (in a non-unique way) as a product $L = S \times \mathbb{R}$, and the metric g_L is given as

$$g_{L(x,t)}((y, \tau), (y, \tau)) = g_{0x}(y, y) + 2\omega_{0x}(y)\tau - \beta(x)\tau^2 \quad (20)$$

for any $(x, t) \in L$, $(y, \tau) \in T_x S \times \mathbb{R}$, where g_0 is a Riemannian metric on S , ω_0 and β are, respectively, a smooth vector field and a smooth positive function on S , and, moreover $Y = \partial_t$.

In what follows, Y will be a prescribed complete stationary vector field in a distinguishing spacetime so that the splitting (20) holds, and the effect of changing the slice S in this splitting will be taken explicitly into account. So a piecewise smooth causal curve $z(s) = (x(s), t(s))$ is future-pointing (resp. past-pointing) if and only if $\beta\dot{t} - \omega_0(\dot{x}) > 0$ (resp. $\beta\dot{t} - \omega_0(\dot{x}) < 0$).

Projections on S of lightlike geodesics of a standard stationary spacetime are pregeodesics for a Randers metric. Indeed, a lightlike curve $z(s) = (x(s), t(s))$, parametrized on a given interval, say $s \in I = [a, b]$, satisfies

$$g_0(\dot{x}(s), \dot{x}(s)) + 2\omega_0(\dot{x}(s))\dot{t}(s) - \beta(x(s))\dot{t}^2(s) = 0 \quad (21)$$

Taking into account the zeros in $\dot{t}$ of this equation, define the Finsler metric

$$F(y) = \left((\omega_0(y)^2 + \beta g_0(y, y))^{1/2} + \omega_0(y) \right) \frac{1}{\beta}, \quad (22)$$

for all $y \in TS$, as well as its reverse metric $\tilde{F}$. According to [18], these Finsler metrics are called *Fermat metrics*. Notice that F is of Randers type, $F = \sqrt{h} + \omega$ with:

$$h(y, y) = \frac{1}{\beta^2} \omega_0(y)^2 + \frac{1}{\beta} g_0(y, y) \quad (23)$$

$$\omega(y) = \frac{1}{\beta} \omega_0(y) \quad (24)$$

for $(x, y) \in TS$. Now, from (21), we have two possibilities for $\dot{t}$:

$$\dot{t} = F(\dot{x}) \quad \dot{t} = -\tilde{F}(\dot{x}), \quad (25)$$

the first equality if z is future-pointing, and the second one if it is past-pointing. Then, putting, $z(a) = (p, t_p)$ the *arrival time* of the curve z , that is, the value of the t coordinate at $z(b)$, is given by:

$$T(z) = t_p + \int_a^b F(\dot{x}) ds \quad \tilde{T}(z) = t_p - \int_a^b \tilde{F}(\dot{x}) ds \quad (26)$$

depending, resp., on if z is future or past-pointing.

The Fermat principle states that z is a critical point of the (future or past) arrival time if and only if z is a (future or past) lightlike pregeodesic (i.e. a geodesic up to a reparametrization) for the spacetime, see [18]. However, it is obvious from (26), that these critical curves coincide with the pregeodesics for F and $\tilde{F}$. Choosing an appropriate parametrization we have finally:

Proposition 3.1. *Let (L, g_L) be a standard stationary spacetime. A curve of the type $z(t) = (x(t), t) \in L$, $t \in [t_0, t_1]$, is a future-pointing, lightlike pregeodesic if and only if $x(t)$, $t \in [t_0, t_1]$, is a unit geodesic for the Fermat metric F defined by (22).*

An analogous statement holds for past-pointing lightlike geodesics and geodesics of the Randers metric $\tilde{F}$.

However, the relation between stationary spacetimes and Randers metrics is much deeper and, in particular, involves the full causal structure of the spacetime [20]. Recall that given two points (*events*) $w, z \in L$, w is *causally related* to z ($w \leq z$) if either $w = z$ or there exists a future-pointing, causal curve from w to z . The *causal future* of $w \in L$ is the set $J^+(w) = \{z \in L \mid w \leq z\}$. An analogous definition holds substituting future-pointing curves with past-pointing ones, so obtaining the *causal past* of w , $J^-(w)$. Spacetimes can be classified according to their increasingly better causal properties getting the so called *causal ladder* of spacetimes (see [8, 52]). In particular, a spacetime is *causal* when it does not contain any closed causal curve, *causally simple* when it is causal and, for any $w \in L$, the causal futures and pasts $J^\pm(w)$ are closed, and *globally hyperbolic* when it is causal and $J^+(w) \cap J^-(z)$ is compact for all w, z (for these definitions, recall [12]). Among other properties, one has:

Theorem 3.2. [20] *Let $L = (S \times \mathbb{R}, g_L)$ be a standard stationary spacetime and (S, F) be its associate Randers space as in (22). Then:*

- (1) *L is causally simple if and only if the space (S, F) is convex.*
- (2) *L is globally hyperbolic if and only if the closed symmetrized balls of the space (S, F) are compact.*

That is, the weakening of the global hyperbolicity condition into causal simplicity for a stationary spacetime is parallel to the weakening of the compactness of the closed symmetrized balls into convexity for (S, F) .

Remark 3.3. As suggested above, the standard stationary splitting is not uniquely determined by, say, the timelike Killing vector field Y . In fact, it can be changed by replacing the spacelike hypersurface S by a new one S' . Such a S' can be written as a (spacelike) graph $S' = \{(x, f(x))\}$ for some function f (whose differential has F -norm smaller than 1). The Fermat metric F' associated to S' satisfies $F' = F - df$ (with natural identifications, see details in [20, Prop. 5.9]). That is, the metric h remains invariant (in fact, h is the metric induced from the orthogonal distribution to ∂_t , up to the conformal factor $1/\beta$, and the one form ω is “gauge transformed” as $\omega' = \omega - df$).

As emphasized in [20], the hypotheses in Theorem 3.2 are invariant under such a change, as they are related to the conformal geometry of the spacetime and, then, they are independent of the choice of the standard splitting. Because of this same reason, the conditions to be studied here are typically independent of the change $\omega \mapsto \omega - df$. In fact, this is obvious in Proposition 2.5 and in formula (14), as only conditions on $d\omega$ (and not on ω itself) are involved.

3.2. Standard stationary domains and light-convexity. Now, choose a domain D of S with smooth boundary ∂D and consider the *stationary domain* $D \times \mathbb{R}$ as a domain of $M = S \times \mathbb{R}$ with boundary $\partial D \times \mathbb{R}$. The Finslerian notions of convexity for the boundaries of domains can be extended to the Lorentzian case, and we can speak on the *infinitesimal* or *local convexity* of $\partial D \times \mathbb{R}$ (recall Remark 2.1 and 2.2). However, in the Lorentzian context it is natural to take into account the causal tripartition of the tangent vectors. Even more, also the structure of standard domains for stationary spacetimes will be taken into account here. So,

consider a function $\Phi : S \times \mathbb{R} \rightarrow \mathbb{R}$, $\Phi(x, t) = \phi(x)$ such that

$$\begin{cases} \Phi^{-1}(0) = \partial D \times \mathbb{R} \\ \Phi > 0 \\ d\Phi(z) \neq 0 \end{cases} \quad \begin{array}{l} \text{on } D \times \mathbb{R} \\ \\ \text{for every } z \in \partial D \times \mathbb{R}. \end{array} \quad (27)$$

We say that $\partial D \times \mathbb{R}$ is *infinitesimally time-convex* (respectively *light-convex*) if for any $z = (x, t) \in \partial D \times \mathbb{R}$ and for any timelike (respectively lightlike) vector $(y, \tau) \in T_x \partial D \times \mathbb{R}$, one has $H_\Phi^{g_L}((y, \tau), (y, \tau)) \leq 0$, where $H_\Phi^{g_L}$ denotes the Hessian of Φ with respect to the Lorentzian metric g_L (recall Remark 2.1). Whenever the last inequality is satisfied with strict inequality, we say that $\partial D \times \mathbb{R}$ is *strongly time-convex* (respectively *strongly light-convex*). The infinitesimal light-convexity can be characterized directly in terms of the corresponding Fermat metric as follows.

Theorem 3.4. *Let (L, g_L) be a standard stationary spacetime and let D be a domain of class C^2 of S . Then $(\partial D; F)$ is infinitesimally convex (resp. strongly convex) if and only if $(\partial D \times \mathbb{R}; g_L)$ is infinitesimally light-convex (resp. strongly light-convex).*

Proof. Observe that the metric g_L can be written as

$$g_L((y, \tau), (y, \tau)) = \left(h(y, y) - (\tau - \omega(y))^2 \right) \beta,$$

where h and ω are defined in (23) and (24). Using this expression, we can easily compute the geodesic equations of (L, g_L) . Denoted by $z = z(s) = (x(s), t(s))$ a geodesic of (L, g_L) , its components x and t satisfy the equations

$$\begin{cases} (\dot{t} - \omega(\dot{x}))\beta = \text{const.} := C_z \\ \frac{1}{2} \nabla^h \beta \left(h(\dot{x}, \dot{x}) - (\dot{t} - \omega(\dot{x}))^2 \right) = \nabla_{\dot{x}}^h(\beta \dot{x}) - C_z \widehat{d\omega}(\dot{x}) \end{cases} \quad (28)$$

where $\widehat{d\omega}$ is the $(1, 1)$ -tensor field h -metrically associated to $d\omega$ as in Proposition 2.5 above.

If z is a lightlike geodesic, then $h(\dot{x}, \dot{x}) - (\dot{t} - \omega(\dot{x}))^2 = 0$ and the second equation in (28) becomes

$$\nabla_{\dot{x}}^h \dot{x} = - \frac{h(\nabla^h \beta, \dot{x})}{\beta} \dot{x} + \frac{C_z}{\beta} \widehat{d\omega}(\dot{x}).$$

If z is future-pointing, from the first equation in (25) and in (28) we have

$$\frac{C_z}{\beta} = \dot{t} - \omega(\dot{x}) = \sqrt{h(\dot{x}, \dot{x})}.$$

Hence the equation satisfied by the x component of a future-pointing lightlike geodesic is

$$\nabla_{\dot{x}}^h \dot{x} = - \frac{h(\nabla^h \beta, \dot{x})}{\beta} \dot{x} + \sqrt{h(\dot{x}, \dot{x})} \widehat{d\omega}(\dot{x}). \quad (29)$$

Arguing as above, we can see that the x component of a past-pointing lightlike geodesic satisfies equation (29) with the $-$ sign instead of $+$ in the right hand side. Let $(x_0, t_0) \in \partial D \times \mathbb{R}$ and $(y_0, \tau_0) \in T_{(x_0, t_0)}(\partial D \times \mathbb{R})$ be a lightlike vector. Consider the lightlike geodesic $z = z(s) = (x(s), t(s))$ such that $z(0) = (x_0, t_0)$ and $\dot{z}(0) = (y_0, \tau_0)$. Since $(\Phi \circ z)''(s) = H_\Phi^{g_L}(\dot{z}(s), \dot{z}(s))$ and $\Phi \circ z = \phi \circ x$, by using (29), (10) and recalling also that y_0 is orthogonal to $\nabla \phi(x_0)$, we get

$$H_\Phi^{g_L}((y_0, \tau_0), (y_0, \tau_0)) = H_\phi^h(y_0, y_0) \pm \sqrt{h(y_0, y_0)} d\omega(y_0, \nabla^h \phi),$$

with the $+$ sign if (y_0, τ_0) is future-pointing and the $-$ sign otherwise. Hence the thesis follows from Proposition 2.5. $\square$

Remark 3.5. There are several subtleties to be taken into account:

(1) Consistently with Remark 3.3, the hypotheses of the theorem are invariant under the change of the standard stationary splitting; indeed, if D is changed to the domain $D' = \{(x, f(x)) : x \in D\}$ for some function $f: D \rightarrow \mathbb{R}$, then $D' \times R$ is clearly equal to $D \times R$ and, by Remark 3.3, $\partial D'$ will be infinitesimally convex w.r.t. F' . Moreover, the equivalence between the convexity for F and $\tilde{F}$ of ∂D in Proposition 2.5 (and also of the domain D), is consistent with the notion of light-convexity, which makes no difference between future and past-pointing lightlike vectors.

(2) Consistently with Theorem 3.4 and Example 2.8, the light-convexity of $(\partial D \times \mathbb{R}; g_L)$ is not related to the convexity of $(\partial D; g_0)$. Indeed the Randers metric in Example 2.8 can be regarded as the Fermat metric associated to $(\mathbb{R}^2 \times \mathbb{R}, g_L)$ where $g_L(x^1, x^2, t)((y^1, y^2, \tau), (y^1, y^2, \tau)) = (1 - f(x^1)^2)(y^1)^2 + (y^2)^2 + 2f(x^1)y^1\tau - \tau^2$.

(3) As in the Finsler case, the notion of infinitesimal convexity for a hypersurface of the type $\partial D \times \mathbb{R}$ in a stationary spacetime is a natural extension of the analogous convexity for a hypersurface in a Riemannian manifold. Moreover, the latter notion is trivially extensible to any embedded hypersurface H in any Lorentzian manifold³, not only the stationary ones. Notice that, as the Hessian depends on the Levi-Civita connection rather than on the metric, the inequality remains in the same direction as in the positive-definite case (that is, no change of sign is required for timelike directions). However, as pointed out in the stationary case, this notion of convexity can be weakened according to the causal character of the involved vectors, that is, we say that H (expressed locally as $\phi^{-1}(0)$ for some ϕ as in (2)) is *infinitesimally light-* (resp. *time-*, *space-*) *convex* if $(H_\phi^{g_L})_z(v, v) \leq 0$ for any $(z, v) \in TH$, with v lightlike (resp. timelike, spacelike).

(4) The notion of *local convexity*, explained in Remark 2.2, is also trivially extensible to the Lorentzian case from the Riemannian or Finslerian ones, and its equivalence with infinitesimal convexity can be also proved by transplanting the technique in [3], see [17]. Again, in the general Lorentzian case, we can define also *local time-*, *space-* or *light-convexity* by considering only geodesics of the corresponding type. However, its equivalence with the corresponding infinitesimal notions is subtler, see [17].

Regarding the last point above, recall that lightlike vectors are points in the boundary of the (open) subsets of both, time and spacelike vectors and, indeed, if a hypersurface is infinitesimally time- or space-convex, then it is also infinitesimally light-convex by continuity. But, in principle, we cannot state that it is also locally convex with respect to lightlike geodesics. Moreover, in principle, the proof of the equivalence between local and infinitesimal convexity in [3] cannot be extended to local and infinitesimal lightlike convexity (see [17, Remark 6]). Nevertheless, in the case of a *standard timelike hypersurface* H in a standard stationary spacetime (i.e

³Usually, one has to assume that the hypersurface is also non-degenerate but, since our definition (recall Remark 2.1) does not involve the second fundamental form of H , the non-degeneracy assumption can be dropped (cf. [17, Remark 3]). Clearly, in the non-degenerate case, one recovers the usual condition about the sign of the second fundamental form.

$H = H_S \times \mathbb{R}$, where H_S is a hypersurface in S), Proposition 3.1 and Theorem 3.4 give the equivalence also in the lightlike case.

Corollary 3.6. *Let $(S \times \mathbb{R}, g_L)$ be a standard stationary spacetime, H_S be a C^2 embedded hypersurface in S . The hypersurface $H = H_S \times \mathbb{R}$ in $S \times \mathbb{R}$ is infinitesimally light-convex if and only if it is locally light-convex.*

Proof. The implication to the left follows easily as in the Riemannian setting. To prove that infinitesimal light-convexity implies local light-convexity, for any $x_0 \in H_S$ take a neighborhood U_S and a function $\Phi: U_S \times \mathbb{R} \rightarrow \mathbb{R}$, $\Phi(x, t) = \phi(x)$, $\phi: U_S \rightarrow \mathbb{R}$ (which satisfies (27) with H_S in place of ∂D and $\phi^{-1}((0, +\infty)) \cap U_S$ replacing D), such that $H_{\Phi}^{g_L}((y, \tau), (y, \tau)) \leq 0$, for all $(y, \tau) \in T(H_S \times \mathbb{R})$ with (y, τ) lightlike. By Theorem 3.4, H_S is infinitesimally convex in $U_S \cap H_S$ for the Fermat metric and, then, by [3, Theorem 1.1] (recall also Remark 2.3) it is locally convex in the same neighborhood of x_0 in H_S with respect to the geodesics of both, the Fermat metric in (22) and its reverse metric $\tilde{F}$ (recall Remark 3.5(1)). This means that for each $x \in U_S \cap H_S$ the exponential maps with respect to F and $\tilde{F}$ send the vectors in a neighborhood of the origin in $T_{x_0}H_S$ into $\phi^{-1}((-\infty, 0]) \cap U_S$. From Proposition 3.1, the exponential map of g_L maps future and past-pointing lightlike vectors in a neighborhood of the origin in $T_{(x,t)}(S \times \mathbb{R})$ into $(\phi^{-1}((-\infty, 0]) \cap U_S) \times \mathbb{R}$ so that H is locally light-convex at any point of $(U_S \times \mathbb{R}) \cap H$. $\square$

Remark 3.7. Notice that a notion like light-convexity (or time-convexity, to be developed next) is local and, thus, it makes sense for hypersurfaces invariant by the flow of the Killing Y , even if the spacetime did not split globally as a standard stationary one (in the spirit of [50]). Moreover, as our characterizations are independent of the chosen standard splitting, the results become applicable to this case. The reader can take this into account and extend the applicability of our approach.

3.3. Time-convexity. Randers metrics can be also used to characterize the convexity of the boundary of a (stationary) region of a standard stationary spacetime with respect to timelike geodesics. Actually, time-convexity can be reduced to light-convexity in a suitable one-dimensional higher product manifold (see [18, Subsection 4.3]), such a trick is valid in a much more general setting for any Lorentzian metric [21, 53]. We start by pointing out some technical properties.

Consider a standard stationary spacetime $(L = S \times \mathbb{R}, g_L)$ as in (20). Let $\mathbb{R}_u \times S$ denote the product manifold $\mathbb{R} \times S$ where the subscript u means that the natural metric $+du^2$ is considered on $\mathbb{R}(\equiv \mathbb{R}_u)$. Put $L_1 = (\mathbb{R}_u \times S) \times \mathbb{R} \equiv \mathbb{R}_u \times L$ and denote the usual projections:

$$\begin{aligned} \Pi_S : \mathbb{R}_u \times S &\rightarrow S, & \Pi_u : \mathbb{R}_u \times S &\rightarrow \mathbb{R}_u, \\ \Pi_1 : \mathbb{R}_u \times L &\rightarrow \mathbb{R}_u, & \Pi : \mathbb{R}_u \times L &\rightarrow L. \end{aligned}$$

Now, endow the manifold $L_1 = \mathbb{R}_u \times L$ with the standard stationary Lorentzian metric g_{L_1} defined as

$$g_{L_1} = \Pi_1^* du^2 + \Pi^* g_L. \quad (30)$$

Obviously, a curve $s \mapsto (u(s), x(s), t(s))$ is a geodesic in (L_1, g_{L_1}) iff $s \mapsto z(s) := (x(s), t(s))$ is a geodesic for g_L and $\ddot{u}(s) = 0$. Then, a lightlike geodesic for g_{L_1} parameterized with a constant $\dot{u}(s) =: \ell$ satisfies $g_L(\dot{z}, \dot{z}) = -\ell^2$, so that $z = z(s)$ is an affinely parametrized timelike geodesic of (L, g_L) , provided that $\ell \neq 0$. The

Fermat metric for $(L_1 = (\mathbb{R}_u \times S) \times \mathbb{R}, g_{L_1})$ takes the form⁴:

$$F_\beta = \sqrt{\Pi_S^* h + \frac{\Pi_u^* du^2}{\beta \circ \Pi_S}} + \Pi_S^* \omega = \sqrt{h_\beta} + \omega_1. \quad (31)$$

where h, ω are as in (23), (24), and:

$$h_\beta = \Pi_S^* h + \frac{\Pi_u^* du^2}{\beta \circ \Pi_S}, \quad \omega_1 = \Pi_S^* \omega, \quad \text{on } \mathbb{R}_u \times S.$$

The arrival time of a future-pointing timelike geodesic $z(s) = (x(s), t(s))$, parameterized on $[a, b]$, connecting a point (p, t_p) of $S \times \mathbb{R}$ to a line $l(\tau) = (q, \tau) \in S \times \mathbb{R}$ and such that $g_L(\dot{z}, \dot{z}) = -\ell^2$ is given by

$$T(z) = t_p + \int_a^b \left(\sqrt{h(\dot{x}, \dot{x}) + \frac{\ell^2}{\beta \circ x}} + \omega(\dot{x}) \right) ds. \quad (32)$$

For a given domain D of class C^2 of S we can study the infinitesimal convexity of $\partial(\mathbb{R}_u \times D) = \mathbb{R}_u \times \partial D$ with respect to F_β in (31). Take ϕ as in (2) globally defined on S (Remark 2.2), and set $\phi_1 : \mathbb{R}_u \times S \rightarrow \mathbb{R}$, as $\phi_1(u, x) = \phi(x)$. By Proposition 2.5, $(\mathbb{R}_u \times \partial D; F_\beta)$ is convex if and only if

$$H_{\phi_1}^{h_\beta}((v, y), (v, y)) + \sqrt{h_\beta((v, y), (v, y))} d\omega_1((v, y), \nabla^{h_\beta} \phi_1) \leq 0, \quad (33)$$

for all $(u, x) \in \mathbb{R}_u \times \partial D$ and $(v, y) \in \mathbb{R}_u \times T_x \partial D$. Trivially, $d\omega_1((v, y), \nabla^{h_\beta} \phi_1) = d\omega(y, \nabla^h \phi)$. Moreover, as $(\mathbb{R}_u \times D, h_\beta)$ is a warped product, taking into account geodesic equations in this kind of manifolds (see e.g. [55, Ch.7, Proposition 38]) and by using (5), it is not difficult to evaluate the Hessian of ϕ_1 with respect to h_β obtaining

$$H_{\phi_1}^{h_\beta}((v, y), (v, y)) = H_\phi^h(y, y) - \frac{h(\nabla^h \phi, \nabla^h \beta)}{2\beta^2} v^2.$$

Summing up, substituting these expressions in (33), one has:

Lemma 3.8. *$(\mathbb{R}_u \times \partial D; F_\beta)$ is infinitesimally convex if and only if*

$$H_\phi^h(y, y) - \frac{v^2}{2\beta^2} h(\nabla^h \phi, \nabla^h \beta) + \sqrt{h(y, y) + \frac{v^2}{\beta}} d\omega(y, \nabla^h \phi) \leq 0 \quad (34)$$

for any $y \in T\partial D$, $v \in \mathbb{R}$.

Likewise the case of lightlike geodesics, the following result holds.

Theorem 3.9. *Let $(S \times \mathbb{R}, g_L)$ be a standard stationary spacetime and let D be a domain of class C^2 of S . Then $(\partial D \times \mathbb{R}; g_L)$ is infinitesimally time-convex (resp. strongly time-convex) if and only if $(\mathbb{R}_u \times \partial D; F_\beta)$ is infinitesimally convex (resp. strongly convex).*

⁴Notice that, for lightlike geodesics, the construction of the Fermat metric was conformally invariant and, so, the elements h, ω were normalized so that β could be regarded as an overall conformal factor, eventually equal to 1. However, this conformal invariance does not hold for timelike geodesics, and it is emphasized by means of the subscript β .

Proof. Let us check that (34) holds if and only if the Lorentzian Hessian of Φ is non-positive on timelike vectors on the tangent bundle of $\partial D \times \mathbb{R}$. To this end we argue as in the proof of Theorem 3.4, with (L_1, g_{L_1}) replacing (L, g_L) . This time since $z = z(s) = (x(s), t(s))$ is timelike, the second equation in (28) becomes

$$-\frac{\nabla^h \beta}{2\beta} v^2 = \nabla_{\dot{x}}^h(\beta \dot{x}) - C_z \widehat{d\omega}(\dot{x}) = \beta \nabla_{\dot{x}}^h \dot{x} + h(\nabla^h \beta, \dot{x}) \dot{x} - C_z \widehat{d\omega}(\dot{x}), \quad (35)$$

where $-v^2 = g_L(\dot{z}, \dot{z}) \neq 0$. As z is future-pointing $\dot{t} = \omega(\dot{x}) + \sqrt{h(\dot{x}, \dot{x}) + \frac{v^2}{\beta}}$ and then $C_z/\beta = \sqrt{h(\dot{x}, \dot{x}) + \frac{v^2}{\beta}}$. Recalling that $\Phi \circ z = \phi \circ x$,

$$H_{\Phi}^{g_L}(\dot{z}(s), \dot{z}(s)) = (\phi \circ z)''(s) = H_{\phi}^h(\dot{x}(s), \dot{x}(s)) + h(\nabla^h \phi(x(s)), \nabla_{\dot{x}(s)}^h \dot{x}(s)).$$

So, computing $\nabla_{\dot{x}}^h \dot{x}$ from (35) the left-hand side of (34) is equal to $H_{\Phi}^{g_L}(\dot{z}(0), \dot{z}(0))$. $\square$

Remark 3.10. The previous result yields a chain of equivalences which, in particular, shows the equivalence between the infinitesimal and local time-convexities for $(\partial D \times \mathbb{R}, g_L)$. In fact, from the construction of g_{L_1} above, $(\partial D \times \mathbb{R}, g_L)$ is locally time-convex iff $(\mathbb{R}_u \times \partial D \times \mathbb{R}, g_{L_1})$ is locally light-convex. By Corollary 3.6, this holds iff $(\mathbb{R}_u \times \partial D \times \mathbb{R}, g_{L_1})$ is infinitesimally light-convex, and by Theorem 3.4, iff $(\mathbb{R}_u \times \partial D; F_{\beta})$ is infinitesimally convex. Finally, by Theorem 3.9 this holds iff $(\partial D \times \mathbb{R}, g_L)$ is infinitesimally time-convex, as required.

In order to apply Theorem 3.9, the following characterization is useful. We emphasize that it is independent of the choice of the standard stationary splitting, in agreement with Remarks 3.3 and 3.5(1).

Proposition 3.11. *Consider a Randers space (S, R) as in (6) and, for any function $\beta > 0$, the Randers space $(\mathbb{R}_u \times S, R_{\beta})$ where R_{β} is constructed as F_{β} in (31). Let D be a domain of class C^2 of S . Then $(\mathbb{R}_u \times \partial D; R_{\beta})$ is infinitesimally convex if and only if the following three conditions hold for all $x \in \partial D$:*

- i) $(\partial D; R)$ is infinitesimally convex, i.e. (Prop. 2.5),

$$H_{\phi}^h(y, y) + \sqrt{h(y, y)} |d\omega(y, \nabla^h \phi)| \leq 0,$$

- ii) $\nabla^h \beta$ does not point outside D at x , i.e.

$$0 \leq h_x(\nabla^h \phi, \nabla^h \beta), \quad (36)$$

- iii) for each $y \in T\partial D$, either

$$d\omega(y, \nabla^h \phi)^2 + \frac{h(\nabla^h \phi, \nabla^h \beta)}{\beta} H_{\phi}^h(y, y) \leq 0 \quad (37)$$

or $h(\nabla^h \phi, \nabla^h \beta) > 0$ and

$$2H_{\phi}^h(y, y) + \frac{\beta}{h(\nabla^h \phi, \nabla^h \beta)} d\omega(y, \nabla^h \phi)^2 + \frac{h(\nabla^h \phi, \nabla^h \beta)}{\beta} h(y, y) \leq 0. \quad (38)$$

Proof. Recall that the convexity of $(\mathbb{R}_u \times \partial D; R_{\beta})$ is equivalent to (34), and put:

$$\lambda^2 = \frac{v^2}{\beta}, \quad r^2 = h(y, y), \quad a r^2 = H_{\phi}^h(y, y), \quad b = \frac{h(\nabla^h \phi, \nabla^h \beta)}{\beta}, \quad dr = d\omega(y, \nabla^h \phi)$$

All these elements except d remain invariant if y is changed by $-y$. So, define the functions $f_{\pm}: [0, +\infty) \times [0, +\infty) \rightarrow \mathbb{R}$,

$$f_{\pm}(r, \lambda) = ar^2 - \frac{b}{2}\lambda^2 \pm dr(r^2 + \lambda^2)^{1/2}.$$

Then, (34) holds if and only if

$$f_+(r, \lambda) \leq 0 \quad \text{and} \quad f_-(r, \lambda) \leq 0 \quad (39)$$

for all $r, \lambda \geq 0$. Evaluating these inequalities at $\lambda = 0$, one has $r^2(a \pm d) \leq 0$, which shows the necessity of i) and gives also

$$a \leq 0 \quad \text{and} \quad d^2 \leq a^2. \quad (40)$$

Evaluating the same inequalities at $r = 0$, we have

$$0 \leq b, \quad (41)$$

which proves the necessity of the condition ii). So, assuming that (41) holds, the conditions (39) are equivalent to

$$d^2 r^2 (r^2 + \lambda^2) \leq \left(ar^2 - \frac{b}{2}\lambda^2 \right)^2,$$

that is:

$$0 \leq (a^2 - d^2)r^4 - (d^2 + ab)r^2\lambda^2 + \frac{b^2}{4}\lambda^4. \quad (42)$$

Finally, under the previous necessary conditions (40)-(41), equation (42) holds iff either its roots, as a polynomial in the variable r^2 are non-positive (i.e. $d^2 \leq -ab$, in agreement with (37)), or if its discriminant, is non-positive (this can happen only if $h(\nabla^h \phi, \nabla^h \beta) \neq 0$ and, under (36), it is equivalent to (38)), as required. $\square$

As an application of the previous result to be applied later, recall:

Corollary 3.12. *Consider a geodesically asymptotically flat end $E^{(k)}$ of a Randers space (see (14)) and a ball $D_{r_0}^{(k)}$ as in (15). The boundary of the domain $(\mathbb{R}_u \times D_{r_0}^{(k)}; R_\beta)$ is strongly convex for large r_0 if β satisfies, as $|x| \rightarrow \infty$:*

$$\beta = C_1 + O(1/|x|^{q'}), \quad \partial_i \beta = O(1/|x|^{q'+1}), \quad \partial_r \beta \sim -\frac{C_2}{|x|^{q'+1}}, \quad (43)$$

for some $C_1, C_2 > 0$, and $q' \in [0, 2q)$, where ∂_r is the vector field $\partial_r = \frac{x^i}{|x|} \partial_i$

Proof. As the Randers manifold is asymptotically flat, from Prop. 2.9 $(\partial D_{r_0}^{(k)}; R)$ is infinitesimally convex, i.e. i) of Proposition 3.11 is satisfied. Recalling (18), we get

$$h(\nabla^h \phi_{r_0}^{(k)}, \nabla^h \beta) = h^{ij} \partial_i \beta \partial_j \phi_{r_0}^{(k)} = -2|x| \partial_r \beta + O(1/|x|^{p+q'}), \quad (44)$$

hence (36) is satisfied for $|x|$ large enough. From (14), (43) and (44), we get

$$\begin{aligned} & \beta d\omega(y, \nabla^h \phi)^2 + H_\phi^h(y, y) h(\nabla^h \phi, \nabla^h \beta) \leq \\ & \leq \frac{C_3 |x|^2 |y|^2}{|x|^{2q+2}} + 4|y|^2 |x| \partial_r \beta + \frac{C_4 |x| |y|^2}{|x|^{p+1}} \left| 2|x| \partial_r \beta + O(1/|x|^{p+q'}) \right| \\ & \leq |y|^2 \left(\frac{C_3}{|x|^{2q}} - \frac{2C_2}{|x|^{q'}} + \frac{C_5}{|x|^{p+q'}} + \frac{C_6}{|x|^{2p+q'}} \right) < 0, \end{aligned}$$

for all $y \in TS_\rho$, that is (37) is also satisfied, for all $y \in T_x \partial D_{r_0}^{(k)}$, provided that $|x| = r_0$ is large enough. $\square$

4. APPLICATIONS TO ASYMPTOTICALLY FLAT STATIONARY SPACETIMES

4.1. The notion of asymptotically flat stationary spacetime. As an application of the results in Subsection 2.4 and of Proposition 3.11 we will consider in the next subsection spheres of large radius in the spacelike slice S of an asymptotically flat stationary spacetime and, in particular, of the stationary region of the Kerr spacetime. But, previously, the notion of asymptotically flat spacetime is revisited now in the framework of stationary spacetimes.

Roughly speaking, for asymptotically flat spacetimes the curvature becomes negligible at large distances from some region, so that the geometry becomes Minkowskian there. This is commonly expressed by assuming the existence of suitable asymptotic coordinates, so that the difference between the original metric and Minkowski one (plus their first and second derivatives) falls-off at an enough fast radial rate. Penrose conformal boundary [57] allows to circumvent the problem of suitably defining and evaluating limits with a truly coordinate-free definition of asymptotic flatness, as done explicitly by Geroch [34]. This intrinsic procedure succeeded (see for example [39, 67, 33]) but, at any case, the appropriate fall-off behavior in coordinates must be recovered at some step.

So, in the particular case of stationary spacetimes, the usual definition of asymptotic flatness implies the existence of a standard stationary splitting with respect to some spacelike hypersurface S such that for some compact set $K \subset S$, $S \setminus K$ is a disjoint union of ends, $E^{(k)}$, $k = 1, \dots, m$, each one admitting asymptotic coordinates $x = (x^1, \dots, x^n)$ where:

$$\begin{aligned} |h_{ij} - \delta_{ij}| + |x||\partial_l h_{ij}| + |x|^2|\partial_{kl}^2 h_{ij}| &= O(1/|x|^\alpha), \\ |\omega_j| + |x||\partial_l \omega_j| + |x|^2|\partial_{kl}^2 \omega_j| &= O(1/|x|^\alpha), \\ |\beta - 1| + |x||\partial_l \beta| + |x|^2|\partial_{kl}^2 \beta| &= O(1/|x|^\alpha), \end{aligned} \tag{45}$$

with $\alpha > 1/2$ (see [10, p.13–14] and also [9]).

In the case of standard static spacetimes, the integrability of the orthogonal distribution $Y^\perp$ to the static vector Y selects a (positive definite) Riemannian manifold –so that the spacetime notion of asymptotic flatness is simplified into the more elementary notion of asymptotic flatness for Riemannian manifolds. Therefore, in this setting, the definition becomes satisfactory, and it is used systematically for problems relative to positive mass and the Riemann-Penrose conjecture (see [14, 15] and references therein).

Hereafter, as only properties of the geodesics will be required, we will not need to impose any bound for derivatives of order greater than 1. Moreover, according to (14), the rate of fall-off at infinity can be arbitrarily slow. As in Subsection 2.4, we will add the surname “geodesically” in the definition in order to distinguish our scarcely restrictive bounds from the more usual ones⁵ (45).

Definition 4.1. *Let $L = S \times \mathbb{R}$ be a standard stationary spacetime with prescribed Killing vector field $Y = \partial_t$ as in (20).*

⁵Recall that, for example, in order to have a well defined, unique and non-necessarily vanishing ADM mass of a 3 dimensional Riemannian manifold (S, h) , the decay rate of h – which involves also the Ricci tensor – must be of order not less than 1/2 and not greater than 1, [2, Theorems 4.2 and 4.3]).

L is geodesically conformally asymptotically flat if its associated Randers space (S, R) is geodesically asymptotically flat, that is, if h and ω in (23) and (24) satisfy (14) in some asymptotic coordinates, for some $p, q > 0$.

In this case, L is geodesically asymptotically flat if β satisfies in asymptotic coordinates:

$$\beta = 1 + O(1/|x|^{q'}), \quad \partial_i \beta = O(1/|x|^{q'+1}), \quad (46)$$

for some $q' > 0$.

Consistently with Remark 3.3, this notion concerns truly geometric elements defined on the spacelike section S which are independent on the chosen standard splitting: the norm of Y , i.e $\beta = -g(Y, Y)$, the metric $h = (g_0/\beta) + \omega^2$ and the cohomology class of ω (the latter univocally determined on the chosen S by the one-form metrically associated to the orthogonal projection of Y on TS). In the particular case of static spacetimes, it is irrelevant if the spacelike hypersurface S is chosen or not orthogonal to the static vector field. On the other hand, notice that, in principle, the notion does depend on the choice of Y ; implicitly, one assumes that the stationary vector field with a better behaviour at infinity is chosen. This is natural for our subsequent study, as we are going to study hypersurfaces invariant by the flow of one such Y (it would not seem reasonable to impose invariance by a second one and, in any case, no better results would be obtained). However, the fact that, in the end, Y will behave asymptotically as a (unit) parallel timelike vector field in Lorentz-Minkowski, suggests that perhaps the dependence of Definition 4.1 with Y could be dropped.

Let us now derive some growth conditions on the metric coefficients of a given standard stationary splitting implying geodesical asymptotic flatness in the sense of Definition 4.1.

Theorem 4.2. *Consider a standard stationary spacetime $(S \times \mathbb{R}, g)$ that satisfies in each end $E^{(k)}$:*

$$\begin{aligned} |(g_0)_{ij} - \delta_{ij}| + |x||\partial_t(g_0)_{ij}| &= O(1/|x|^{p_0}), \\ |(\omega_0)_j| + |x||\partial_t(\omega_0)_j| &= O(1/|x|^{q_0}), \\ |\beta - 1| + |x||\partial_t\beta| &= O(1/|x|^{q'}), \end{aligned} \quad (47)$$

for some $p_0, q_0, q' > 0$. Then $(S \times \mathbb{R}, g)$ is geodesically asymptotically flat (with $p = \min\{p_0, 2q_0, q'\}$ and $q = q_0$).

Proof. We recall that

$$h_{ij} = \frac{(g_0)_{ij}}{\beta} + \frac{(\omega_0)_i(\omega_0)_j}{\beta^2} \quad \text{and} \quad \omega_i = \frac{(\omega_0)_i}{\beta};$$

hence,

$$h_{ij} = \frac{\delta_{ij} + O(1/|x|^{p_0})}{1 + O(1/|x|^{q'})} + \frac{O(1/|x|^{2q_0})}{(1 + O(1/|x|^{q'}))^2} = \delta_{ij} + O(1/|x|^{\min\{p_0, 2q_0, q'\}}).$$

Moreover,

$$\begin{aligned}
 \partial_k h_{ij} &= \\
 & - \frac{\partial_k \beta}{\beta^2} (g_0)_{ij} + \frac{1}{\beta} \partial_k (g_0)_{ij} - \frac{2\partial_k \beta}{\beta^3} (\omega_0)_i (\omega_0)_j + \frac{1}{\beta^2} (\partial_k (\omega_0)_i (\omega_0)_j + (\omega_0)_i \partial_k (\omega_0)_j) \\
 & = O(1/|x|^{q'+1}) + O(1/|x|^{p_0+1}) + O(1/|x|^{q'+1+2q_0}) + O(1/|x|^{2q_0+1}) \\
 & = O(1/|x|^{\min\{p_0, q'\}+1}) + O(1/|x|^{2q_0+1}).
 \end{aligned}$$

Arguing analogously, we get:

$$\partial_k \omega_i = O(1/|x|^{q_0+q'+1}) + O(1/|x|^{q_0+1}) = O(1/|x|^{q_0+1}).$$

Therefore, defining $p = \min\{p_0, 2q_0, q'\}$ and $q = q_0$, we see that h and ω satisfies (14) and then, since β satisfies (46) by assumptions, (L, g) is geodesically asymptotically flat. $\square$

Remark 4.3. It is worth to observe that the addition of further fall-off hypotheses for the second derivatives, as in (45), would not be so innocent as it seems. Recall that such fall-off hypotheses are commonly used; in fact, in order to define asymptotic flatness by means of the approach based on Penrose conformal embeddings and boundaries, one uses commonly the simplifying hypothesis that the spacetime is vacuum (Ricci-flat) in a neighborhood of the asymptotic boundary $\mathcal{J}^\pm$ (see for example [33, 39, 66, 67])⁶. For a vacuum stationary solution of the Einstein equations, there exist coordinates in which the metric is analytic [54]. Moreover, for an asymptotically flat and vacuum stationary solution there exists a system of coordinates in a neighborhood of infinity where $h_{ij} - \delta_{ij}$, ω_i , $\beta - 1$ decay as $1/|x|$ and each of their derivatives of order k , as $1/|x|^{1+k}$, for any $k \in \mathbb{N}$ (see [23, Sect. 2.3] and [24, Theorem 2.1, Eq. (26) and Lemma 2.5]).

Nevertheless, surprising difficulties appear in the stationary non-static case, because of the implications of the fall-off of curvature. First, recall that very few examples of exact solutions modeling *vacuum and rotating* isolated objects in general relativity are presently known. The list of useful solutions presently consists of the Neugebauer-Meinell dust (a rigidly rotating thin disk of dust with finite radius surrounded by an asymptotically flat vacuum region), and a few variants, see [45, 46]. Moreover, as emphasized by Roberts [60], *there is no known perfect fluid source which can be matched to a Kerr vacuum exterior*, as one would expect in order to create the simplest possible model of a rotating star⁷—this contrasts with the plenitude of solutions which match to Schwarzschild. So, the true applicability of the results in this contexts requires accurate hypotheses, as the optimized fall-off hypotheses in our definition.

Finally, it is also interesting to check further the consistency of our notion of geodesic *conformal* asymptotic flatness, with the classical one obtained by using Penrose conformal embeddings. On the one hand, in the (conformally) vacuum

⁶Even though this excludes the presence of electromagnetic fields, it is justified as it allows one to make simpler statements, leaving to the reader the task to determine precise fall-off hypotheses for the energy-momentum tensor (compare, e.g., with [33, Section 2.3]) whenever the vacuum assumption is dropped.

⁷See, about the difficulty of this problem, the claim on its over-determinacy in [48], as well as the construction of a solution involving Kerr-Newman spacetime [47] (recall that the latter is not vacuum around $\mathcal{J}^\pm$).

case the classical notion of such asymptotic flatness (as explained, for example, in [33, Sect. 2.3]) imply the existence of a double cone structure (essentially, two copies of $S^2 \times \mathbb{R}$) for the points of the conformal boundary which are accessible from the spacetime by means of causal curves. As emphasized in [27], there are reasons to prefer the recently revisited notion of *causal boundary* to the conformal one, as the former is a (explicitly intrinsic) general construction, applicable even when no useful conformal embedding is known—but, under quite general hypotheses, it agrees with the conformal boundary when this can be defined. In the case of stationary spacetimes, the stationary-to-Randers correspondence allows to characterize the causal boundary as a double cone structure on the Busemann boundary of the associated Randers manifold [28]; in particular, it agrees with Penrose's in the classical vacuum case (see further details in [26]). On the other hand, in the classical notion, the Weyl tensor goes to 0 on the boundary. As in the case of the Riemann tensor, our mild fall-off requirements in Defn. 4.1 are tailored for geodesics and do not imply such a behavior—but, obviously, our definition and results are applicable if such an additional hypotheses is imposed, as in Weyl conformally flat vacuum solutions [65, Chapter 20].

4.2. Large spheres in the spacelike slice of an end. Next, let us apply the results on convexity to some hypersurfaces in a geodesically asymptotically flat spacetime. As in Subsection 2.4, we will consider large spheres $S^{n-1}(r_0)$ defined in the coordinates of each end as $|x|^2 = r_0^2$ ((regarded as the boundary of the region where $|x|^2 < r_0^2$, except if otherwise specified). As a direct consequence of Proposition 2.9 and Theorem 3.4:

Corollary 4.4. *For any geodesically conformally asymptotically flat spacetime, all the hypersurfaces $S^{n-1}(r_0) \times \mathbb{R}$ are strongly light-convex for any sphere $S^{n-1}(r_0)$ of large radius r_0 (i.e., with r_0 greater than some constant).*

However, time-convexity is subtler, as large balls will not fulfil this property in all asymptotically flat spacetimes. In fact, recall first the following straightforward consequence of Theorem 3.9 and Corollary 3.12.

Corollary 4.5. *Let $L = S \times \mathbb{R}$ be a geodesically asymptotically flat standard stationary spacetime. If, in addition to (46), β satisfies*

$$\partial_r \beta \sim -\frac{C}{|x|^{q'+1}} \quad (48)$$

being⁸ $q' \in (0, 2q)$ and $C > 0$, then the hypersurfaces $S^{n-1}(r_0) \times \mathbb{R}$ are strongly time-convex for any sphere $S^{n-1}(r_0)$ of large radius r_0 .

Remark 4.6. The additional condition on $\partial_r \beta$ in Corollary 4.5 is crucial for time convexity, as it can be easily understood from Lorentz-Minkowski spacetime. Recall that the hypersurfaces $S^{n-1}(r_0) \times \mathbb{R}$ in $\mathbb{L}^n$ are time-convex, but not strongly time convex. In fact, any line $x_0 \times \mathbb{R}$ with $x_0 \in S^{n-1}(r_0)$, regarded as a timelike geodesic, remains in the boundary of the domain and does not leave it. So, the sign of $\partial_r \beta$ will be crucial, as the next corollary will make apparent. However, (48) is not satisfied by physically reasonable asymptotically flat spacetimes, as discussed below (see Proposition 4.9).

⁸The result follows even if we allow here and in (46) $q' = 0$.

Corollary 4.7. *Consider an asymptotically flat spacetime as in Theorem 4.2; let $C > 0$ be a constant and let $S^{n-1}(r_0)$ be a sphere of radius r_0 in $E^{(k)}$. Then the following properties hold:*

(1) *if $q' \in (0, 2q_0)$ and*

$$\partial_r \beta \sim -\frac{C}{|x|^{q'+1}}, \quad \text{as } |x| \rightarrow +\infty,$$

then the hypersurfaces $S^{n-1}(r_0) \times \mathbb{R}$ are strongly time-convex for any r_0 large enough;

(2) *if*

$$\partial_r \beta \sim \frac{C}{|x|^{q'+1}}, \quad \text{as } |x| \rightarrow +\infty,$$

then, for any r_0 large enough, the hypersurfaces $S^{n-1}(r_0) \times \mathbb{R}$ are (strongly light-convex but) not infinitesimally time-convex.

Proof. The first part follows from Theorem 4.2 and Corollary 4.5. For the second part, we first observe that for a large enough r_0 the hypersurfaces $S^{n-1}(r_0) \times \mathbb{R}$ is light-convex by Theorem 4.2 and Corollary 4.4. Moreover, considering $\phi = \phi(r) = r_0^2 - r^2$, we get

$$\begin{aligned} h(\nabla^h \phi, \nabla^h \beta) &= -2|x|\partial_r \beta + O(1/|x|^p)|x| \left(\sum_k (\partial_k \beta)^2 \right)^{1/2} \\ &< -2|x| \frac{C}{2|x|^{q'+1}} + O(1/|x|^p)|x| O(1/|x|^{q+1}) \\ &= -C/|x|^{q'} + O(1/|x|^{p+q'}), \end{aligned}$$

which is negative for r_0 large enough. Thus, Eq. (36) is not satisfied on $S^{n-1}(r_0)$ and Proposition 3.11 plus Theorem 3.9 implies that the hypersurfaces $S^{n-1}(r_0) \times \mathbb{R}$ are not time-convex for any r_0 large enough. $\square$

Remark 4.8. Observe that, as a difference with Corollaries 4.4 and 4.5, where only assumptions on the first derivatives of ω are considered, in Theorem 4.2 and in Corollary 4.7, also conditions on the asymptotic behaviour of the one-form ω_0 are imposed ((47) are indeed used). Nevertheless, such asymptotic conditions for ω_0 are not necessary if we consider an appropriate combination with the behaviour of g_0 , in the spirit of Definition 4.1. Recall that if one changes the standard stationary splitting by considering a slice $S' = \{(x, f(x))\}$ as in Remark 3.3, the new standard stationary spacetime $(S' \times \mathbb{R}, g)$ remains, of course, geodesically asymptotically flat and large spheres in any end $(E')^{(k)} = \{(x, f(x)) | x \in E^{(k)}\}$, will produce hypersurfaces in $S' \times \mathbb{R}$ time-convex or not according to Corollary 4.7. This is because, as already recalled in Remark 3.3, the metric h^f on S' is isometric to h , β is invariant and the one-form ω^f on S' is the push-forward, by the map $x \in S \mapsto (x, f(x)) \in S'$, of $\omega - df$.

In the light of Corollary 4.7, it becomes important to give conditions in order to understand the sign of the constant C in the asymptotic behaviour of β . Under natural physical hypotheses, this coefficient is the Komar's mass and, so, its sign is expected to be non-negative [22, p. 462]. More precisely, following Choquet-Bruhat [22, Defn. 9.1, p. 55], a 4-spacetime will be called *Einsteinian* when it satisfies Einstein equations with reasonable conditions for the matter. This definition is

consciously ambiguous. For an asymptotically flat spacetime, here *Einsteinian* will mean just: (a) it fulfils Definition 4.1 with $p = q = q' = 1$ (i.e., the natural power for the asymptotic decay holds), (b) either g_0 is complete or the Komar mass in a r -bounded region that includes the removed compact subset K is nonnegative (i.e., at each end, the integral of Komar form $\star dY^b$ on a compact surface enclosing K is nonnegative) and, (c) in the particular case that Komar mass of an end is 0, positive mass theorem is applicable (recall that Komar and ADM masses are expected to be equal [22, Th. 4.13]), so that the spacetime is $\mathbb{L}^4$. Other natural physical assumptions will be stated explicitly in the following result, which follows from [22, Theorem 4.11, p.462] and Corollary 4.7.

Proposition 4.9. *Let $(S \times \mathbb{R}, g) \neq \mathbb{L}^4$ be an Einsteinian stationary spacetime. Assume that, at each end, there exists a constant C such that*

$$\partial_r \beta = C/|x|^2 + o(1/|x|^2), \quad \text{as } |x| \rightarrow +\infty.$$

Moreover, assume that the Ricci tensor $\text{Ric}(g)$ of the metric g satisfies

$$\text{Ric}(g)(\partial t, \partial t) \geq 0,$$

and the function $\text{Ric}(g)(\partial t, \partial t)$ is integrable on S . Then, at each end, C must be positive and, for any r bigger than some large $r_0 > 0$, the hypersurface $S^{n-1}(r) \times \mathbb{R}$ is not infinitesimally time-convex.

Remark 4.10. In the next subsection, we will study Kerr spacetime as a paradigmatic example and will check directly the behaviour of β ensured by Proposition 4.9. But such an asymptotic behaviour holds for any physically reasonable asymptotically flat spacetime, as discussed above (see also [10, §3.1]).

4.3. Convex shells in Kerr spacetime. We will focus now on Kerr spacetime. We will check that, even though large asymptotic spheres are strongly light-convex, *they are not infinitesimally time-convex*, and take advantage of the intuition on this spacetime in order to interpret physically such a result.

Recall that in Boyer-Lindquist coordinates (r, θ, φ, t) the metric of the Kerr spacetime is given by

$$ds^2 = -\frac{\Delta}{\rho^2} (dt - a \sin^2 \theta d\varphi)^2 + \frac{\sin^2 \theta}{\rho^2} ((r^2 + a^2)d\varphi - adt)^2 + \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\theta^2,$$

$$\Delta = r^2 - 2mr + a^2, \quad \rho^2 = r^2 + a^2 \cos^2 \theta,$$

where m is the ADM mass and $a = j/m$ with j the ADM angular momentum of the spacetime. The above metric is standard stationary outside the stationary limit hypersurface $\mathcal{N}$ (ergosurface $r = m + \sqrt{m^2 - a^2 \cos^2 \theta}$), which appears in *slow* Kerr spacetime (i.e., the black hole model where $a^2 < m^2$). So, by Theorem 4.2, it is immediate to check that this region is a geodesically asymptotically flat stationary spacetime; thus, Proposition 2.9 is applicable and from Theorem 3.4, the submanifolds $S^2(r_0) \times \mathbb{R}$ are strongly light-convex for r_0 large enough. For time-convexity, recall that, from the expressions above,

$$\beta = \frac{r^2 - 2mr + a^2 \cos^2 \theta}{r^2 + a^2 \cos^2 \theta} = 1 - \frac{2mr}{r^2 + a^2 \cos^2 \theta},$$

Thus, (2) of Corollary 4.7 holds and the submanifolds $S^2(r_0) \times \mathbb{R}$ are not time-convex for r_0 large enough.

The same results hold also in the stationary region of the Kerr-Newman spacetime. In fact, the metric of the Kerr-Newman solution is obtained from the Kerr one by replacing Δ with $\Delta + q^2$, where q is the electric charge of the spacetime. Therefore, it is geodesically asymptotically flat and, since

$$\beta = 1 - \frac{2mr - q^2}{r^2 + a^2 \cos^2 \theta},$$

(2) of Corollary 4.7 is satisfied as well.

Remark 4.11. These results admit the following natural physical interpretation. First, let us consider the non-time convexity of the large asymptotic balls. Assume that a pebble is tossed straight upward. If it lacks escape energy, the gravitational attraction will make the pebble reach a maximum value r_0 and fall down. The trajectory of the pebble will be then a timelike geodesic which violates the local time-convexity of the sphere of radius r_0 . More precisely, any timelike geodesic in Kerr(-Newman) spacetime with a r -turning point $r_0 = r(s_0)$ which is a strict local maximum of ⁹ $r(s)$, violates the time-convexity of $S^2(r_0) \times \mathbb{R}$. However, it is easy to realize that such a behaviour does not hold if, instead of a pebble, a light-ray is emitted (as light-rays initially propagating both, radially outwards and tangent to a large sphere, will attain increasing arbitrarily big values of r_0), so that light-convexity is always achieved.

As a final digression, we emphasize that, in the computations above, only the stationary region of the spacetime is being considered. For example, in the Kerr spacetime, such a region lies outside the ergosurface $\mathcal{N}$. In this hypersurface the Killing vector ∂_t is lightlike ($\beta = 0$) and the induced metric on $\mathcal{N}$ is degenerate at the poles $\theta = 0, \pi$. So, it cannot be regarded as a (timelike) boundary $\partial D \times \mathbb{R}$ of a domain in the same sense as above. Even though the notions of time and light convexity would make still sense, new subtler possibilities would appear. The final applications on connecting geodesics are expected to hold with independence of the details of this boundary (see Remark 5.6). Nevertheless, for the sake of completeness, we make some computations about the convexity of hypersurfaces close to $\mathcal{N}$. Following [49, Sect. 7.2], we consider the domain $M_\varepsilon^a := D_\varepsilon^a \times \mathbb{R}$, for each $\varepsilon > 0$, where

$$D_\varepsilon^a = \{(x, y, z) \in \mathbb{R}^3 : m + \sqrt{m^2 + \varepsilon - a^2 \cos^2 \theta} < r\}. \quad (49)$$

These domains M_ε^a exhaust $M^a := M_{\varepsilon=0}^a$ when $\varepsilon \searrow 0$ and, in the case of Schwarzschild spacetime (i.e. $a = 0$), they are just spheres with radius $r_\varepsilon = 2m + \varepsilon$ (greater than Schwarzschild radius $r = 2m$). Observe that in the Schwarzschild spacetime, being $\omega \equiv 0$, the Fermat metric reduces to the Riemannian metric $h = (\frac{1}{\beta} dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\varphi^2)^{\frac{1}{\beta}}$ where now $(\beta \equiv) \beta(r) = 1 - 2m/r$. The function ϕ_ε , defining the sphere of radius r_ε as its 0-level set and assuming positive values on the domain $D_\varepsilon = \{(x, y, z) \in \mathbb{R}^3 : r_\varepsilon < r\}$, is $\phi_\varepsilon(r) = r^2 - r_\varepsilon^2$. Thus the gradient (with respect to

⁹Following the detailed discussion and nomenclature about Kerr in [56, p.209], such geodesics are either (ordinary) *bounded orbits* or (exceptional) *crash-crash* ones. Recall that the ordinary orbits of Kerr geodesics are *bounded* (those with two r -turning points, one of them a maximum), *flyby* (one r -turning point, necessarily a minimum) or *transit* (no r -turning points). The exceptional orbits are *spherical* ones (i.e. $r(s) \equiv r_0$), asymptotic orbits to a spherical one, *crash-escape* orbits and *crash-crash* ones. None of these have a r -turning point r_0 except the crash-crash one (as in the example of the pebble), where r_0 is a maximum

h) of β does not point outside D_ϵ (i.e. (36) is satisfied) and (34) becomes equivalent to the infinitesimal convexity of the sphere ∂D_ϵ with respect to h . For Schwarzschild spacetime, the Christoffel symbols of h involved in the computation of $H_{\phi_\epsilon}^h$ on $T\partial D_\epsilon$ are:

$$\Gamma_{\theta\theta}^r = m - r_\epsilon\beta, \quad \Gamma_{\theta\varphi}^r = 0, \quad \Gamma_{\varphi\varphi}^r = (m - r_\epsilon\beta) \sin^2 \theta$$

hence, from Theorems 3.4 and 3.9, $\partial D_\epsilon \times \mathbb{R}$ is strongly light-convex and strongly time-convex if $m - r_\epsilon\beta > 0$, i.e. if $0 < \epsilon < m$ (observe that strong light-convexity seems to fail at $\theta = 0, \pi$ but these values must not be taken into account because belong to the boundary of the open subset where spherical coordinates are injective; clearly by rotating the r semi-axis, also the poles on the sphere of radius r_ϵ corresponding to $\theta = 0, \pi$ can be covered). For $\epsilon = m$, we have $m - r_\epsilon\beta = 0$ and then $\partial D_{\epsilon=m} \times \mathbb{R}$ is strongly time-convex but only light-convex. For arbitrary values of a , the shape of the ergosurface changes and such convexities are not expected. However, for ϵ and $|a|$ sufficiently small, Kerr regions M_ϵ^a are strongly time- and light-convex too. A simple proof of this fact (simplifying and slightly improving the one in [49, Prop. 7.2.1]) follows recalling that, in this case, M_ϵ^a can be regarded as a small perturbation of $M_\epsilon^{a=0}$ and Kerr metric as a small C^2 perturbation of Schwarzschild one, when a goes to 0. By the strong light-convexity of $M_\epsilon^{a=0}$, any of (7)-(9) is satisfied with a strict inequality. As $\omega \equiv 0$ and $\partial D_\epsilon^{a=0}$ is compact, the h -Hessian of ϕ_ϵ is less than a strictly negative constant, thus the same must hold for ∂D_ϵ^a in the Kerr spacetime, provided that a is small enough (to this end, notice that in the Kerr metric $\omega = -(2mra \sin^2 \theta d\varphi)/(\rho^2 - 2mr)$ (see, for example, [36]) and its derivatives goes to 0 when a tends to 0, uniformly on ∂D_ϵ). For strong time-convexity we can reason in a similar way, because, as already observed above, (34) holds with strict inequality on the boundary of Schwarzschild spheres. Nevertheless, space-convexity does not hold even for small $|a|$ (including $a = 0$) and small $\epsilon > 0$, see [29, Corollary 2].

Putting together the previous discussion plus Corollary 4.4, we get the following result about light-convexity of some shells in slow Kerr spacetime:

Corollary 4.12. *Let $m > 0$ and $a \in \mathbb{R}$ such that $a^2 < m^2$. For any $\epsilon \in (0, m)$, and $r_0 \in (0, \infty]$, let D_{ϵ, r_0}^a be the following subset of $\mathbb{R}^3$:*

$$D_{\epsilon, r_0}^a = \{(x, y, z) \in \mathbb{R}^3 : m + \sqrt{m^2 + \epsilon - a^2 \cos^2 \theta} < r < r_0\}.$$

Then there exists $a_0 > 0$, depending on ϵ , such that the stationary domain $M_{\epsilon, r_0}^a := D_{\epsilon, r_0}^a \times \mathbb{R}$ of Kerr spacetime M^a has strongly light-convex boundary for each $|a| < a_0$, provided that r_0 is large enough.

5. APPLICATIONS TO TOPOLOGICAL LENSING AND CAUSAL SIMPLICITY

5.1. Lightlike geodesics. In [18, Proposition 4.1], using Proposition 3.1, it was proved that if (S, F) is forward or backward complete then any point $w = (p, t_p)$ of L can be joined to a flow line $l = l(\tau) = (q, \tau)$ of the Killing field ∂_t by means of a past-pointing lightlike geodesic γ and, moreover, if S is non-contractible, a sequence $\{\gamma_m\}$ of such geodesics with negatively diverging arrival times exists. Clearly, from Proposition 3.1, the existence of at least one such a geodesic is an immediate consequence of the completeness assumption on (S, F) and of the Hopf-Rinow theorem in Finsler geometry (cf. [1, Theorem 6.6.1]). From an infinite-dimensional variational

viewpoint, the multiplicity result is obtained applying Lüsternik–Schnirelmann theory to the energy functional of the Finsler metric. Indeed, if S is non-contractible, the Lüsternik–Schnirelmann category of the manifold $\Omega_{p,q}(S)$ of H^1 -paths between the points p and q on S is infinite, [25], and then the energy functional, which satisfies the Palais–Smale condition on $\Omega_{p,q}(S)$, [18], must admit infinitely many critical points, whose energy goes to infinity, and then infinitely many geodesics between p and q . If p and q lie on the same closed geodesic, the geodesics connecting p to q might be the multiple coverings of the closed one, as happens for two non-antipodal points on a sphere; on the other hand, even in this case, the lightlike geodesics on $S \times \mathbb{R}$ arising from the Finsler ones on S have distinct supports (while the projections on S of the supports coincide). Notice that this multiplicity result admits a natural interpretation of *topological lensing* of the connecting geodesics. In fact, light rays emitted at different moments in the past from the stellar object, represented by l , will arrive at different directions for the observer at the single event w .

In [20], the above mentioned result in [18] was reinterpreted and improved. Concretely, the completeness of (S, F) was substituted by the weaker assumption that the closed balls $\bar{B}_s(p, r)$ are compact. In fact, this condition becomes equivalent to the global hyperbolicity of L (recall Theorem 3.2(2)) and, in such a spacetime, any two causally related points can be joined by means of a length-maximizing causal geodesic (Avez–Seifert result). As a consequence, the existence of the arrival-time maximizing, past-pointing lightlike geodesic γ follows directly from purely causal grounds (just realizing that, because of stationarity, $J^-(p)$ must intersect any flow line l of Y). The variational technique is required only for the existence of the sequence $\{\gamma_m\}$ and, as also shown in [20], the assumption on $\bar{B}_s(p, r)$ is enough for this purpose. Recall that, the existence and multiplicity of connecting light rays is then equivalent to the existence and multiplicity of geodesics connecting two points in a Randers manifold. The latter problem is well understood in Finsler geometry (see for example [1]) and can appear because of topological reasons (as explained above) or just by curvature focalization (existence of conjugate points for the Randers manifold). So, the existence of gravitational lensing is completely characterized from this viewpoint.

Next, we can extend such results to domains D of S having convex boundary. This will be a consequence of the previous results combined with the results in [3] where, first, the equivalence between different notions of convexity for the boundary ∂D of a domain D in a Finsler manifold was proven and, then, the convexity of D is characterized as follows (see [3, Theorem 1.3]):

Theorem 5.1. *Let D be a C^2 domain¹⁰ of a smooth manifold M endowed with a Finsler metric F having $C_{\text{loc}}^{1,1}$ fundamental tensor (see (1)) and such that the intersection of the closed symmetrized balls $\bar{B}_s(p, r)$, $p \in D$, with $\bar{D}$ is compact. Then, D is convex if and only if ∂D is convex.*

Moreover, in this case, if additionally D is not contractible, then any pair of points in D can be joined by infinitely many connecting geodesics contained in D and having diverging lengths.

¹⁰Recall that [3, Theorem 1.3] is stated under $C_{\text{loc}}^{2,1}$ regularity, but the result holds also for a C^2 domain (see [17]).

Remark 5.2. From the technical viewpoint, an improvement of the previous result will become relevant later. Recall that we can consider also the distance $d_s^{\bar{D}}$ obtained by computing the distances (the Finslerian distance d and then the symmetrized one d_s) by using only curves contained entirely in $\bar{D}$. This distance is intrinsic to $\bar{D}$, i.e., it is independent of the extension of F outside. Obviously, $d_s^{\bar{D}}$ is greater or equal to the restriction of d_s to $\bar{D}$ and, so, the corresponding balls satisfy $\bar{B}_s^{\bar{D}}(p, r) \subset \bar{B}_s(p, r) \cap \bar{D}$. Then, if $\bar{B}_s(p, r)$ is compact so is $\bar{B}_s^{\bar{D}}(p, r)$, but the converse does not hold (see Example 5.9 below). By checking the proof of Theorem 5.1 in [3], it is not hard to prove:

All the conclusions of Theorem 5.1 hold if the hypothesis on the compactness for $\bar{B}_s(p, r)$, is replaced by the more general hypothesis of compactness for the intrinsic balls $\bar{B}_s^{\bar{D}}(p, r)$, $p \in D$.

In fact, the compactness of the sets $\bar{D} \cap \bar{B}_s(p, r)$ is used in Lemma 4.2 of [3] and in the proof of the Palais-Smale condition for the functionals of a family perturbing the energy functional of the Finsler manifold (see [3, Prop. 4.3]), to ensure that the supports of any sequence of absolutely continuous curves $\gamma_n: [0, 1] \rightarrow D$, $\gamma_n(0) = p$, $\gamma_n(1) = q$, $p, q \in D$, such that

$$\int_0^1 F^2(\dot{\gamma}_n) ds \leq C,$$

for a constant $C > 0$ independent of n , are contained in a compact subset of M . The same holds if the intrinsic balls $\bar{B}_s^{\bar{D}}(p, r)$, $p \in D$, are compact. In fact

$$d^{\bar{D}}(p, \gamma_n(s)) \leq \int_0^s F(\dot{\gamma}_n) d\tau \leq \left(\int_0^1 F^2(\dot{\gamma}_n) ds \right)^{\frac{1}{2}} \leq C^{1/2}$$

and analogously $d^{\bar{D}}(\gamma_n(s), q) \leq C^{1/2}$, hence the supports of the curves γ_n are contained in the subset $\bar{B}_s^{\bar{D}}(p, C^{1/2} + \frac{d^{\bar{D}}(q, p)}{2})$ of $\bar{D}$.

Our main result for lightlike geodesics is then:

Theorem 5.3. *Let $L = S \times \mathbb{R}$ be a C^2 spacetime endowed with a $C_{\text{loc}}^{1,1}$ standard stationary Lorentzian metric g_L , and let D be a C^2 domain of S . Consider on S the Fermat metric F defined in (22) (see also (6), (23) and (24)), and assume that any intrinsic closed symmetrized ball $\bar{B}_s^{\bar{D}}(p, r)$, $p \in D$, is compact (which happens, in particular, if any $\bar{B}_s(p, r) \cap \bar{D}$ is compact). Then, the following assertions are equivalent:*

- (1) $(D \times \mathbb{R}, g_L)$ is causally simple.
- (2) (D, F) is convex.
- (3) $(\partial D; F)$ is convex (infinitesimally or, equivalently, locally, [3, Corollary 1.2]).
- (4) $(\partial D \times \mathbb{R}; g_L)$ is light-convex (infinitesimally or, equivalently, locally, recall Corollary 3.6).
- (5) Any point $w = (p, t_p) \in D \times \mathbb{R}$ and any line $l_q := \{(q, \tau) \in D \times \mathbb{R} : \tau \in \mathbb{R}\}$, with $p \neq q$, can be joined in $D \times \mathbb{R}$ by means of a future-pointing lightlike geodesic $z(s) = (x(s), t(s))$ which minimizes the (future) arrival time T in (26) (equivalently, such that x minimizes the F -distance in D between p and q).

- (6) Any w and l_q as above can be joined in $D \times \mathbb{R}$ by means of a past-pointing lightlike geodesic $z(s) = (\tilde{x}(s), \tilde{t}(s))$ which maximizes the (past) arrival time $\tilde{T}$ in (26) (equivalently, such that $\tilde{x}$ minimizes the reverse F -distance in D between p and q).

In this case, if D is not contractible, then sequences of both, future-pointing and past-pointing lightlike geodesics joining $w = (p, t_p)$ and l_q contained in $D \times \mathbb{R}$ and with diverging arrival times, exist for all $p, q \in D$.

Proof. For (1) $\Leftrightarrow$ (2) apply Theorem 3.2(1); for (2) $\Leftrightarrow$ (3), Theorem 5.1 (plus Remark 5.2); for (3) $\Leftrightarrow$ (4) Theorem 3.4. Finally, (5) (or (6)) $\Leftrightarrow$ (2) follows from Proposition 3.1, recalling also (26) (notice also that (5) $\Leftrightarrow$ (1) holds in a more general context, see Remark 5.4(2) below). The multiplicity result is a direct consequence of the last statement in Theorem 5.1. $\square$

Remark 5.4. (1) In other results on the connection of w and l_q with a lightlike geodesic [32], both, the light-convexity of $\partial D \times \mathbb{R}$ and some global assumptions on the growth of ω and β in a standard stationary splitting are required as sufficient conditions. In comparison, our result is optimal because: (a) it characterizes the light-convexity of $\partial D \times \mathbb{R}$ in terms of the convexity of (D, F) , and (b) it gives the natural global hypotheses (i.e., the compactness of $\bar{B}_s^{\bar{D}}(p, r)$ —or at least of $\bar{B}_s(p, r) \cap \bar{D}$) which, as it can be easily showed, is always implied by the global assumptions in the previous results.

The ambient hypothesis (compactness of $\bar{B}_s^{\bar{D}}(p, r)$ in (b)) is not strictly necessary, but it appears clearly as a natural hypothesis —not just as a technical assumption. In fact, even for a Riemannian manifold, if the boundary of a domain D is supposed to determine the convexity of the domain, one assumes typically that the domain is included in a complete Riemannian manifold. This hypothesis can be weakened by the more intrinsic assumption that the closed balls in $\bar{D}$ are compact. Nevertheless, no more true generality is obtained then: when the last hypothesis holds (either in the version that the closed balls of $\bar{D}$ are intrinsic or in the one that they are the intersection of balls in S with $\bar{D}$ are compact), one can modify the original Riemannian metric outside $\bar{D}$ so that it becomes complete¹¹. In the stationary/Finsler case, however, we retain explicitly the weakest hypothesis, due to the subtleties of the symmetrized distance (plus the existence of other elements in the full spacetime).

Our optimal result was known for the special case of standard static spacetimes ($\omega_0 \equiv 0$ in (20)) which is simpler, as the associated Randers space becomes indeed a Riemannian one (see [6, Proposition 5.1]).

(2) The appearance of causal simplicity in Theorem 5.3 can be seen from the following general viewpoint. Consider any causally simple spacetime L , a point $p \in L$ and an inextensible future-directed causal curve l , which enters in $J^+(p)$ at some point $q = l(t_0)$. Necessarily, $q \in \overline{J^+(p)} \setminus I^+(p)$ and, as $J^+(p)$ is closed, q belongs to the horizon $J^+(p) \setminus I^+(p)$. Then, necessarily there exists a future-directed

¹¹One could try to improve this hypothesis by replacing it with some condition intrinsic to D (i.e, independent of if it is regarded as a domain of a bigger manifold) or relaxing the hypotheses of smoothness on ∂D . Nevertheless, such conditions are rather technical even in the Riemannian case [5] (and, thus, even in the simple static case). Moreover, the non-trivial relation between the symmetrized distance d_s and the generalized distance d in a Finsler manifold, complicates more the situation —recall, for example, that d_s may not be a length metric, which must be taken into account in the picture of the intrinsic Cauchy boundary, see [28, Remark 3.23].

lightlike geodesic γ from p to $q(=l(t_0))$. Moreover, γ minimizes the “proper arrival time” of the observer whose world-line is a reparametrization of l , in the sense that t_0 is the minimum value of the parameter t of l such that p can be connected with $l(t)$ by means of a future-directed causal curve. In fact, a causal spacetime L is causally simple if and only if for any point p and any future-directed (resp. past-directed) causal curve l which enters $J^+(p)$ (resp. $J^-(p)$), there exists a first point $l(t_0)$ such that $l(t_0) \in J^+(p)$ (resp. $l(t_0) \in J^-(p)$) —which necessarily can be joined with p by means of a lightlike geodesic.

As a direct consequence of Theorem 5.3 and Corollary 4.12, we get:

Corollary 5.5. *The regions M_{ϵ, r_0}^a of Kerr spacetime (see Corollary 4.12) are causally simple provided that r_0 is large enough and, for each ϵ , $|a| \geq 0$ is small enough. Thus, in this case, for any point $w = (p, t_p)$ and any line $l_q(\tau) = (q, \tau)$ in M_{ϵ, r_0}^a , both, a sequence of future-pointing and one of past-pointing lightlike geodesics connecting w and l_q with diverging arrival times, exist.*

Existence and multiplicity results as the ones in Corollary 5.5 were already known for a region of the type $M_{\epsilon, \infty}^a$ [32, 49]. They can be also obtained for some shells strictly contained in the domain of outer communication and intersecting the ergosphere, in a slow or extreme Kerr-Newman spacetime, [38].

Remark 5.6. For the physical applications of the results along all Section 5, the following observations are in order. First, when the spacetime L is globally hyperbolic, the technical assumption that the intrinsic symmetrized closed balls $\bar{B}_s^D(p, r)$ are compact, is automatically satisfied for any domain D (recall Theorem 3.2 (2) and Remark 5.2) Whenever L models a physical isolated body with no black holes, L is assumed to be globally hyperbolic and geodesically asymptotically flat and, thus, our results are applicable —i.e., the existence of connecting geodesics inside large balls is assured, and criteria to estimate their minimum radius are also provided. In the case that the spacetime contains a black hole, L would represent only its stationary part and the applicability of the results to large balls depends also on the (time, light) convexity of the stationary limit hypersurface $\mathcal{N}$, where the causal character of Y changes. If L was all the domain of outer communication¹² (as happens in Schwarzschild spacetime) L is expected to be globally hyperbolic and all the results would be applicable again. Otherwise (as happens in slow Kerr, where $\mathcal{N}$ is the limit of the ergosphere), the results may be still applicable in some cases, as shown in Corollary 5.5. And, at any case, the domain of outer communication would include L and would have as a boundary some Killing horizon H (by Hawking’s theorem, under some technical hypotheses, see [23, Sect. 3.1.1] and references therein). H would play the role of a light (and time) convex boundary, as it is a degenerate hypersurface foliated by lightlike geodesics. Therefore, the results on connecting geodesics are expected to hold (even though such geodesics might eventually cross the ergosphere for $Y = \partial_t$, as in [38]).

5.2. Timelike geodesics. A further application of Theorem 5.1, concerns timelike geodesics in standard stationary spacetimes.

Consistently with the notations of Subsection 3.3, let d^{F_β} be the generalized distance on $\mathbb{R}_u \times S$ determined by $F_\beta = \sqrt{h_\beta} + \omega_1$ and $d_s^{F_\beta}$ be the corresponding symmetrized distance. For each $(u_0, p_0) \in \mathbb{R}_u \times S$, let $\bar{B}_s((u_0, p_0), r)$ denote the

¹²We use standard terminology as in [23].

closure of the r -ball in $\mathbb{R}_u \times S$ with respect to $d_s^{F_\beta}$. As before, $\bar{B}_s(p_0, r)$ denote the closure of the r -ball in S with respect to the symmetrized distance d_s^F associated to the generalized distance d^F on S (given as $F = \sqrt{h} + \omega$), and $\bar{B}_s^{\bar{D}}(p, r)$ denotes the intrinsic r -ball in $\bar{D}$ obtained from the intrinsic distance $d_s^{\bar{D}}$ (Remark 5.2).

Our aim is to prove the following result:

Theorem 5.7. *Let $L = S \times \mathbb{R}$ be a C^2 spacetime endowed with a $C_{\text{loc}}^{1,1}$ standard stationary Lorentzian metric g_L and let D be a C^2 domain of S . Assume that all the intrinsic closed symmetrized balls $\bar{B}_s^{\bar{D}}(p, r)$, $p \in D$ are compact (which happens, in particular, if all $\bar{B}_s(p, r) \cap \bar{D}$ are compact). Then:*

- (A) $(\mathbb{R}_u \times \partial D; F_\beta)$ is convex (infinitesimally or, equivalently, locally, [3, Theorem 1.1]) if and only if for any $\ell > 0$, any point $w = (p, t_p) \in D \times \mathbb{R}$ and any line $l_q(\tau) = (q, \tau) \in D \times \mathbb{R}$, $\tau \in \mathbb{R}$, can be joined by a future-pointing timelike geodesic $z(s) = (x(s), t(s)) \in D \times \mathbb{R}$, $s \in I = [0, 1]$, with Lorentzian length ℓ such that x minimizes the arrival time $t(1)$ among all the future-pointing causal curves from p to q of the same length ℓ .
- (B) If $(\mathbb{R}_u \times \partial D; F_\beta)$ is convex and D is not contractible, then a sequence of future-pointing timelike geodesics $z_n(s) = (x_n(s), t_n(s))$, $s \in I = [0, 1]$, having Lorentzian length ℓ , joining w and $l(\mathbb{R})$, having support in $D \times \mathbb{R}$ and diverging arrival times (32) exists.

Before proving this theorem, the following comments are in order.

Remark 5.8. (1) From the technical viewpoint, it is worth pointing out that the compactness of the subsets $\bar{B}_s(\bar{p}_0, r_0) \cap \bar{D}$, $p_0 \in D, r_0 > 0$, does not imply the compactness of the analogous subsets $\bar{B}_s((u_0, p_0), r) \cap (\mathbb{R}_u \times \bar{D})$ in $\mathbb{R}_u \times S$ (even in the case when $\omega = 0$), see Example 5.9 below. Nevertheless, if we consider instead the intrinsic balls, the analogous property will hold (Lemma 5.10 below). Because of this reason, the improvement in Remark 5.2 of Theorem 5.1 becomes important here.

(2) The idea of the proof will be to reduce the problem to the lightlike case, by using the metric g_{L_1} explained in Subsection 3.3 and taking into account the previous point (1). So, the assertions in (A) can be also refined taking into account Theorem 5.3. In particular, analogous statements hold for past-pointing timelike geodesics and the hypothesis that $(\mathbb{R}_u \times \partial D; F_\beta)$ is convex in (A) can be replaced by any of the following alternatives:

- (i) $(\mathbb{R}_u \times D; F_\beta)$ is convex (recall Theorem 5.1);
- (ii) the boundary of $(\mathbb{R}_u \times D \times \mathbb{R}, g_{L_1})$ is light-convex (infinitesimally or, equivalently, locally, Corollary 3.6);
- (iii) $(\partial D \times \mathbb{R}; g_L)$ is time-convex (infinitesimally or, equivalently, locally, recall Remark 3.10);
- (iv) $(\mathbb{R}_u \times D \times \mathbb{R}, g_{L_1})$ is causally simple (Theorem 5.3, applied to the standard stationary spacetime $(\mathbb{R}_u \times D \times \mathbb{R}, g_{L_1})$).

Moreover, the convexity of $(\mathbb{R}_u \times \partial D; F_\beta)$ is directly computable from Prop. 3.11.

(3) As a consequence, the causal simplicity of $(\mathbb{R}_u \times D \times \mathbb{R}, g_{L_1})$ implies the causal simplicity of $(D \times \mathbb{R}, g_L)$; notice that the converse does not hold. Such a property is general for any spacetime (L, g) (easily, if $(\mathbb{R}_u \times L, g_1 = du^2 + g)$ is causally simple then so is (L, g) , see [51, Th. 3.6] for a more general result). In the

stationary case, this is parallel to the fact that the time-convexity for a boundary implies light-convexity, but the converse does not hold.

Example 5.9. Consider the standard static spacetime with $S = \mathbb{R}$ and $\beta = e^{x^2}$, i.e., $L = \mathbb{R}^2, g_L = -e^{x^2}dt^2 + dx^2$ so that the Fermat metric is $F = \sqrt{h}$ with $h = dx^2/e^{x^2}$ and the balls for F and h agree. Now,

$$(L_1, g_{L_1}) = (\mathbb{R}_u \times \mathbb{R}^2, g_{L_1} = du^2 + dx^2 - e^{x^2}dt^2), \\ \mathbb{R}_u \times S = \mathbb{R}^2, \quad h_1 = e^{-x^2}(du^2 + dx^2), \quad F_\beta = \sqrt{h_1}.$$

Notice that $(\mathbb{R}^2, h_1)$ has a finite diameter $R_0 (< \infty)$ (say, $d^{h_1}((0,0), (u_M, 0)) < 1 + 2 \int_0^\infty e^{-x^2/2} dx$ for all $u_M \in \mathbb{R}$, as one can take curves starting at $(0,0)$ which go far along the x axis, then move straight in the u -direction until reaching $u = u_M$, and finally come along the x -direction to $(u_M, 0)$). In particular, the $d_s^{F_\beta}$ -ball of radius R_0 is non-compact.

Now, take $D = (-1, 1) \subset \mathbb{R}$. As $\bar{D}$ is compact, the intersection of the closed d^h -balls with $\bar{D}$ are compact, that is, *the subsets $\bar{B}_s(p_0, r_1) \cap \bar{D}$, $p_0 \in D$, $r_1 > 0$ are always compact*. However, *the analogous subsets $\bar{B}_s((u_1, p_1), r_1) \cap (\mathbb{R}_u \times \bar{D})$ in $\mathbb{R}_u \times S$ are not compact* for $r_1 \geq R_0$, as they include all the region $\mathbb{R}_u \times \bar{D}$. Nevertheless, the intrinsic balls $\bar{B}_s^{\mathbb{R}_u \times \bar{D}}((u_1, p_1), r_1)$ are compact (in agreement with the next lemma), as β is bounded on $\bar{D}$.

Lemma 5.10. *Assume that the intrinsic balls $\bar{B}_s^{\bar{D}}(p_0, r_0)$, $p_0 \in D$, $r_0 > 0$ are compact subsets in $\bar{D}$. Then, the intrinsic balls $\bar{B}_s^{\mathbb{R}_u \times \bar{D}}((u_1, p_1), r_1)$ are also compact subsets in $\mathbb{R}_u \times \bar{D}$.*

Proof. It is enough to check that $\bar{B}_s^{\mathbb{R}_u \times \bar{D}}((u_1, p_1), r_1)$ lies in a compact product subset of $\mathbb{R}_u \times \bar{D}$. Let (u_2, p_2) be a point in $\bar{B}_s^{\mathbb{R}_u \times \bar{D}}((u_1, p_1), r_1)$ and let γ^i , $i = 1, 2$, be two curves $\gamma^i: [0, 1] \rightarrow \mathbb{R}_u \times \bar{D}$, $\gamma^i(s) = (u^i(s), x^i(s))$, γ^1 from (u_1, p_1) to (u_2, p_2) and γ^2 from (u_2, p_2) to (u_1, p_1) , such that $\int_0^1 F_\beta(\dot{\gamma}^i) ds < 2r_1$. Hence, for each $s \in [0, 1]$, we have $d^{\bar{D}}(p_1, x^1(s)) < 2r_1$ and $d^{\bar{D}}(x^1(s), p_1) < 4r_1$ (for the latter inequality, it is enough to consider the arc of x^1 from $x^1(s)$ to p_2 and then the curve x^2 from p_2 to p_1). Thus $d_s^{\bar{D}}(p_1, x^1(s)) < 3r_1$ for all $s \in [0, 1]$. As $\bar{B}_s^{\bar{D}}(p_1, 3r_1)$ is compact, there exist positive constants A, B, C , independent of the curve x^1 , such that $\beta(x^1(s)) \leq A^2$ and $|\omega(\dot{x}^1(s))| \leq B|\dot{x}^1(s)|_h \leq CF(\dot{x}^1(s)) \leq CF_\beta(\dot{\gamma}^1(s))$, for all $s \in [0, 1]$. Thus, we have

$$|u_2 - u_1| \leq \int_0^1 |\dot{u}^1| ds \leq A \int_0^1 \left(\frac{(\dot{u}^1)^2}{\beta} + h(\dot{x}^1, \dot{x}^1) \right)^{1/2} ds \\ = A \left(\int_0^1 F_\beta(\dot{\gamma}^1) ds - \int_0^1 \omega(\dot{x}^1) ds \right) \leq 2Ar_1(1 + C) =: K,$$

hence the closed ball $\bar{B}_s^{\mathbb{R}_u \times \bar{D}}((u_1, p_1), r_1)$ is contained in the compact subset $[u_1 - K, u_1 + K] \times \bar{B}_s^{\bar{D}}(p_1, 3r_1)$ of $\mathbb{R}_u \times S$. $\square$

Proof of Theorem 5.7. As described in Subsection 3.3, the existence of future-pointing timelike geodesics having Lorentzian length ℓ and connecting the point (p, t_p) to the line $l_q(\tau) = (q, \tau)$ in the region $D \times \mathbb{R}$ contained in (L, g_L) , is equivalent to the existence of future-pointing lightlike geodesics connecting the point $(0, p, t_p)$ and the line $\tilde{l}_q(\tau) = (\ell, q, \tau)$ in the region $\mathbb{R}_u \times D \times \mathbb{R}$ contained in (L_1, g_{L_1}) (see (30)). In turn, the latter is equivalent, by Proposition 3.1, to the existence of

geodesics, with respect to the Randers metric F_β , connecting the points $(0, p)$ and (ℓ, q) in $\mathbb{R}_u \times D$ and then, from Theorem 5.1, Remark 5.2 and Lemma 5.10, it is equivalent to the convexity of $(\mathbb{R}_u \times \partial D; F_\beta)$. $\square$

Remark 5.11. The applicability of Theorem 5.7 is determined by Proposition 3.11 (and, in particular, Corollary 3.12). For the case of the stationary region M^a of Kerr spacetime, Theorem 5.7 is applicable by choosing D equal to the domain D_ε^a in (49) for any $\varepsilon \in (0, m)$ and any small enough $|a| \geq 0$ (as ∂M_ε^a is then time-convex, as discussed above Corollary 4.12) but not to the domain D_{ε, r_0}^a in Corollary 4.12 (M_{ε, r_0}^a is only light-convex).

The violation of time-convexity (physically interpreted in Remark 4.11) can be analysed by considering connecting timelike geodesics of prescribed length $|v| > 0$ between some fixed p and l_q . Recall that the second and third terms in the right hand side of (34) goes to 0 for large r_0 , while the first one remains of the order $-2|y|^2$, according to (19). So, a geodesic γ which violates time-convexity (in the sense that remains in the closure of M_{ε, r_0}^a but touches the component $S^2(r_0) \times \mathbb{R}$ of its boundary) will have small $|y|$ -component. This corresponds with the component along $TS^2(r_0)$ of $\dot{\gamma}$ and it is related to the angular momentum of γ . Geodesics with $|y|/|v|$ greater than some positive constant will not violate time-convexity for sufficiently large radius r_0 .

Remark 5.12. Notice also that, in cosmological models, the metric is typically *globally conformal* to a stationary one (actually, a static one) and, thus, the techniques are applicable to this case. For example, in a FLRW model, the spacetime is written as a warped product $(I \times S, g = -dt^2 + f(t)^2 \pi_S^* g_S)$ where $I \subset \mathbb{R}$ is an interval, $\pi_S : I \times S \rightarrow S, t : I \times S \rightarrow I$ are the natural projection and (S, g_S) is Riemannian manifold. The conformal metric g/f^2 can be written as a product spacetime $J \times S$, where $J \subset \mathbb{R}$ is another interval determined by the change of variable $ds = dt/f(t)$. To obtain results on connecting lightlike geodesics, it is enough to apply the former ones to the latter product, taking into account that the s -interval covered by the geodesic must be included in J .

5.3. Revision of the techniques from the SRC viewpoint. At the beginning of Subsection 5.1, we explained how the result of existence of at least one lightlike connecting geodesic in manifolds without boundary in [18] could be obtained from purely causal grounds. Next, we will consider the direct implications of causality for manifolds with boundary on the existence of connecting causal geodesics. We will focus on the case of timelike geodesics, as for lightlike ones, the question is simpler (recall Remark 5.4 (2)). We will see how direct techniques of causality plus SRC allow to prove Proposition 5.15 below. Then, we will discuss the different techniques and results.

Let us start with preliminary results on causality. Recall that, as far as causality is only involved, C^1 regularity for the ambient manifold and the domain D , and C^0 for the Lorentzian metric will be enough.

Lemma 5.13. *A product spacetime $(\mathbb{R}_u \times M, g_1 = du^2 + g_L)$ is causally simple iff*

- (i) (M, g_L) is causally simple,
- (ii) *the time-separation d_L for (M, g_L) is continuous between points $w, w' \in M$ with $d_L(w, w') < \infty$ (here, $d_L(w, w') = \sup_{z \in \mathcal{C}(w, w'; M)} \ell_{g_L}(z)$, where $\mathcal{C}(w, w'; M)$ is the set of the future-pointing causal curves z from w to w' , and $\ell_{g_L}(z) = \int_z \sqrt{-g_L(\dot{z}, \dot{z})}$ is the Lorentzian length),*

- (iii) if $w \leq w'$ and $d_L(w, w') < \infty$ then there exists a future-pointing causal geodesic σ from w to w' with length equal to $d_L(w, w')$.

Proof. This is just a particular case of [51, Theorem 3.13], which is stated for any product $H \times M$, (apply it to $H = (\mathbb{R}_u, du^2)$). $\square$

Recall that, in the previous result on connectivity, the finiteness of the time separation d_L becomes essential to ensure both, the existence of connecting causal geodesics and the continuity of d_L . As the second statement of the following result shows, in stationary spacetimes the finiteness of the time separation d of $D \times \mathbb{R}$ (regarded as a spacetime by itself, i.e., $d(w, w') = \sup_{z \in \mathcal{C}(w, w'; D \times \mathbb{R})} \ell_{g_L}(z)$) is ensured by the compactness of the closed symmetrized balls.

Lemma 5.14. *For any stationary domain $(D \times \mathbb{R}, g_L)$:*

- (1) *The time-separation d is unbounded on the stationary lines, i.e. for all $w = (p, t_p) \in D \times \mathbb{R}$ and $q \in D$, $\lim_{\tau \rightarrow \infty} d(w, l_q(\tau)) = \infty$.*
- (2) *If the intrinsic balls $\bar{B}_s^{\bar{D}}(p_0, r)$, $p_0 \in D$, $r > 0$, are compact subsets in $\bar{D}$, then the time-separation d is finite valued on $D \times \mathbb{R}$.*

Proof. (1) Recall that w and l_q can be joined by means of a future-directed lightlike curve $z(s) = (x(s), t(s))$, $s \in [0, 1]$ (choose any curve x in D connecting p and q , and choose t so that z becomes lightlike), and that the length of l_q between $z(1)$ and $l_q(\tau)$ goes to infinity for large τ .

(2) Assume by contradiction that $d((p, t_p), (q, t_q)) = \infty$, and let

$$z_n(s) = (x_n(s), t_n(s)), \quad s \in [0, 1]$$

be a sequence of causal curves in D from (p, t_p) to (q, t_q) with diverging Lorentzian lengths. Any point $z_n(s)$ lies in $J^+((p, t_p)) \cap J^-((q, t_q))$ and, so, all the curves x_n lie in the intersection between the closures of the forward F -ball $B^{\bar{D}^+}(p, t_q - t_p)$ and the backward one $B^{\bar{D}^-}(q, t_q - t_p)$ (use [20, Eq. (4.6)]), and then they are contained in $K := \bar{B}_s^{\bar{D}}(p, t_q - t_p + \frac{d^{\bar{D}}(q, p)}{2})$ which is a compact set. As the Fermat metric $F = \sqrt{h} + \omega$ is positive definite, there exists some $\varepsilon > 0$ and $\eta > 0$ such that the h -norm of ω satisfies $\|\omega\|_x < 1 - \varepsilon$ and $\beta(x) \leq \eta^2$, for all $x \in K$. Then, as the arrival times of all the curves $\{z_n\}$ is t_q , equation (32) implies that the h -length of all the curves x_n is bounded. Then, parameterizing the curves z_n at constant speed gives $\ell_n = \sqrt{-g_L(\dot{z}_n, \dot{z}_n)} = \int_0^1 \sqrt{g_L(\dot{z}_n(s), \dot{z}_n(s))} ds \rightarrow \infty$ and, from (32), we get a contradiction

$$\begin{aligned} t_q - t_p + (1 - \varepsilon) \int_0^1 \sqrt{h(\dot{x}_n, \dot{x}_n)} ds &\geq t_q - t_p - \int_0^1 \omega(\dot{x}_n) ds \\ &= \int_0^1 \sqrt{h(\dot{x}_n, \dot{x}_n) + \frac{v_n^2}{\beta(x_n)}} ds \geq \int_0^1 \sqrt{\frac{\ell_n^2}{\beta(x_n)}} ds \geq \frac{1}{\eta} \ell_n \rightarrow \infty. \end{aligned}$$

$\square$

As a consequence of the previous two lemmas, we can give the following relevant particular case of Theorem 5.7 (recall also Remark 5.8(1)) by using strictly causal hypotheses and proof (including stationary-to-Randers correspondence, SRC).

Proposition 5.15. *Assume that a stationary domain $(D \times \mathbb{R}, g_L)$ satisfies that $(\mathbb{R}_u \times D \times \mathbb{R}, g_{L_1} = \Pi_1^* du^2 + \Pi^* g_L)$ is causally simple and that the intrinsic balls*

$\bar{B}_s^{\bar{D}}(p_0, r)$, $p_0 \in D$, $r > 0$, are compact subsets in $\bar{D}$. Then any point $w = (p, t_p) \in D \times \mathbb{R}$ and any line $l_q(\tau) = (q, \tau) \in D \times \mathbb{R}$, $\tau \in \mathbb{R}$, can be joined by a future-pointing timelike geodesic $z(s) = (x(s), t(s)) \in D \times \mathbb{R}$, $s \in I = [0, 1]$ with Lorentzian length ℓ , such that x minimizes the arrival time $t(1)$ among all the future-pointing causal curves from p to q of the same length ℓ .

Proof. Notice that, as d is finite-valued from Lemma 5.14(2), the assertion (iii) of Lemma 5.13 implies that Avez-Seifert property holds, i.e. each two points $w, w' \in D \times \mathbb{R}$ which are strictly causally related ($w < w'$), can be connected by means of a causal geodesic of length $d(w, w') \in (0, +\infty)$. Let τ_0 be the infimum of the $\tau \geq t_p$ such that $w \leq (q, \tau)$. Clearly, $d(w, (q, \tau_0)) = 0$ (by the assertion (i) of Lemma 5.13, $w \leq (q, \tau_0)$ too) and the property in Lemma 5.14(1) plus the continuity of d (assertion (ii) in Lemma 5.13) imply that the time-separation between w and some point of l_q is ℓ . Thus, Avez-Seifert property yields the result. $\square$

As a consequence, the considerations about the generality of our results in the lightlike case in comparison with the previous ones (Remark 5.4), can be extended to the case of Theorem 5.7. Summing up, our conclusions about the obtained results and required techniques are the following ones.

- (1) In the purely causal proof of Proposition 5.15, SRC has been used to show the finiteness of d (Lemma 5.14), which is crucial for the Avez-Seifert property. In comparison with the results in the previous subsection, the limitations of this causal result 5.15 are:
 - (a) it does not explain when $(\mathbb{R}_u \times D \times \mathbb{R}, g_{L_1} = \Pi_1^* du^2 + \Pi^* g_L)$ is causally simple in terms of the stationary spacetime $(D \times \mathbb{R}, g_L)$, and
 - (b) it does not allow a result on the multiplicity of the connecting timelike geodesics.
- (2) The limitation (a) is again remedied by SRC (Theorem 5.3 applied to the stationary spacetime $(\mathbb{R}_u \times D \times \mathbb{R}, g_{L_1})$). For the limitation (b), an infinite-dimensional variational approach is required¹³. In fact, the result in Theorem 5.1 (plus Remark 5.2) was claimed in the proof of our main Theorem 5.7. The formulation of the hypothesis of convexity with the function ϕ allows to connect the geometric interpretations with the variational approach, which uses a functional with a penalization term constructed from ϕ (see [3, Section 4]).
- (3) At any case, the statement and proof of our main result (Theorem 5.7 plus Remark 5.8) requires both, variational results and causal interpretations, including SRC. This allows to obtain optimal analytical hypotheses. In fact, we use only the overall hypothesis of the compactness of closed symmetrized balls, which becomes natural even in the Riemannian case (as explained in Remark 5.4(1)), and has a role clearly revealed in Lemma 5.14(2)). Up to this ambient hypothesis, our condition for the problem of causal connectedness is both, necessary and sufficient.
- (4) As a consequence, our results improve all the previous references on the topic. Essentially these references were obtained by using variational methods, and achieved sufficient conditions for causal connectedness. In general, typical analytic conditions imply the conditions in Proposition 3.11

¹³However, one could still find a result of multiplicity in purely causal terms by using timelike homotopy classes as in [62], which allows some sharp conclusions on the behavior of the geodesics.

which characterize (together with Theorem 3.9 and Remark 3.10) the time-convexity of $\partial D \times \mathbb{R}$.

In particular, Theorem 5.7 improves the results in [18, Sect. 4.3], by dealing with manifolds with boundary, and those in [35, Th. 1.6, 1.7] by giving the full characterization of causal connectedness with natural geometric interpretations. The results in [4] obtained by means of a different approach based on a relation between geodesics and Lagrangian systems, are also improved. In this reference, the existence of timelike geodesics between a point and a line of the boundary was proved, under time-convexity, only for a suitable range of values for the Lorentzian length ℓ , depending on the metric coefficients.

Remark 5.16. For further developments, the following possibilities are pointed out. First, our approach based on SRC is potentially useful also for other variational problems, such as periodic trajectories or trajectories critical for another (time-independent) functionals, see [7] and references therein.

However, we emphasize that only causal geodesics are studied by using SRC. The problems which include spacelike geodesics, as geodesic connectedness, must introduce also other subtle techniques. It is worth pointing out, in relation with the cases obtained here:

- (a) domains such as $M_\epsilon^a, \epsilon > 0$ in Kerr spacetime (see Subsection 4.3) not only are not space-convex but also are not geodesically connected [29, Corollary 3],
- (b) the full outer region of Schwarzschild spacetime, as well as the outer region of slow Kerr one (outside the black hole, which include the ergosphere and, so, it is not fully stationary for $a \neq 0$) is geodesically connected [30]; the proof uses different topological arguments introduced in [31], and
- (c) for general standard stationary spacetimes, geodesic connectedness has been studied by a combination of variational and causal methods in [16] (see also [17], for the case of domains with boundary).

Even though such problems of geodesic connectedness do not have a simple physical interpretation as those of *causal* geodesic connectedness in this paper, they constitute an excellent arena to study critical points curves for indefinite functionals, with broad possibilities of applications.

6. APPENDIX: CONCLUSIONS ON PHYSICAL APPLICATIONS

The problem of visibility of stellar objects (existence and multiplicity of causal geodesics connecting points and world-lines) under physically reasonable properties has been analysed, being the following ingredients relevant in its solution:

- Relativistic Fermat's principle, as known from the works by Kovner [42] and Perlick [58], asserts that if a connecting causal curve from a point to an observer (world-line) with a critical arrival time for the observer's proper time exists, then it is a lightlike geodesic. However, in order to ensure the existence of such a geodesic, a mixture of variational techniques (critical point theory), topological elements (Ljusternik-Schnirelmann theory) and geometrical equivalences (stationary-to-Randers correspondence) in the framework of Causality theory, has been used. In particular:
 - (i) The existence of connecting causal geodesics is characterized in terms of the geodesic connectedness of the associated Finsler manifolds of Randers

type. Therefore, the notion of convexity (for domains of the spacetime) is analyzed carefully.

(ii) The multiplicity of connecting geodesics (lensing) appears naturally either due to curvature (the geodesics of the associated Fermat metric present conjugate points, a well understood property in Finsler Geometry) or to global topological properties (non-contractibility of the manifold).

- We have considered only stationary spacetimes (or *strictly* stationary in the nomenclature of some references, as we are assuming that the causal character of the Killing vector field must remain timelike on all the spacetime). Nevertheless, the applications to asymptotically flat spacetimes make the results applicable in more general situations which include black holes, and the conformal invariance of most of the techniques make them applicable even to cosmological models.
- It is natural to consider the case of connecting geodesics that are confined in a spherical shell of the spacetime and, so, large balls in asymptotically flat spacetimes have been especially studied. The obtained results show under which circumstances our intuition on known spacetimes as Kerr's one can be transplanted to arbitrary asymptotically flat spacetimes (recall Proposition 4.9).
- The stationary to Randers correspondence allows a better understanding of the notion of asymptotic flatness, connecting it with the asymptotic behaviour of a Finsler manifold and interpreting the role of the cohomology of the shift ω as a significant geometric object in that notion.
- The results include not only lightlike geodesics but also timelike ones. For these geodesics, it is assumed that they arrive in a prescribed lifetime. This becomes natural from both viewpoints, the mathematical one (otherwise, the results on multiplicity would become trivial) and the physical one (the particles might disintegrate).
- The revision of the previous techniques in the literature shows the limitations of each one, as well as the optimality of the obtained results.

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DIPARTIMENTO DI MECCANICA, MATEMATICA E MANAGEMENT
 POLITECNICO DI BARI
 VIA ORABONA 4, 70125, BARI, ITALY
E-mail address: caponio@poliba.it

DIPARTIMENTO DI MATEMATICA
 UNIVERSITÀ DEGLI STUDI DI BARI
 VIA ORABONA 4, 70125 BARI ITALY
E-mail address: germinar@dm.uniba.it

DEPARTAMENTO DE GEOMETRÍA Y TOPOLOGÍA
 FACULTAD DE CIENCIAS, UNIVERSIDAD DE GRANADA
 CAMPUS FUENTENUEVA S/N, 18071 GRANADA, SPAIN
E-mail address: sanchezm@ugr.es