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Semiclassical states for NLS equations with magnetic potentials having polynomial growths

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We prove existence of standing wave solutions for a nonlinear Schrödinger equation on $\mathbb{R}^3$ under the influence of an external magnetic field B. In particular we deal with the physically meaningful case of a constant magnetic field $B=(0,0,b)$ having source in the potential $A(x)=(b/2)(-x_2,x_1,0)$ corresponding to the Lorentz gauge. © *2005 American Institute of Physics.* [DOI: 10.1063/1.1874333]

I. INTRODUCTION

In quantum mechanics the introduction of an external magnetic field $B:\mathbb{R}^3\to\mathbb{R}^3$ involves replacing the gradient operator ∇ with $\nabla+iA(x)$ where A is a vector (or magnetic) potential and satisfies curl $A(x)=B(x)$. The Schrödinger operator with a magnetic field having source in A and a scalar (electric) potential W has the following expression:

$$L^{\hbar}_{A,W}=\left(\frac{\hbar}{i}\nabla-A\right)^2+W(x)=-\hbar^2\Delta-\frac{2\hbar}{i}A\cdot\nabla+|A|^2-\frac{\hbar}{i}\mathrm{div}\,A+W(x), \tag{1}$$

where $i^2=-1$ and $\hbar$ is the Planck constant. We notice that if we replace the magnetic potential A by $\tilde{A}(x)=A(x)+\nabla\varphi(x)$ for some real-valued C^2 function φ then $\tilde{B}(x)=\mathrm{curl}\,\tilde{A}(x)=\mathrm{curl}\,A(x)=B(x)$ and

$$e^{-i\varphi}\left[\left(\frac{1}{i}\nabla-\tilde{A}(x)\right)^2+W(x)\right]e^{i\varphi}=\left(\frac{1}{i}\nabla-A(x)\right)^2+W(x),$$

so that the spectral properties of $L^{\hbar}_{A,W}$ and $L^{\hbar}_{\tilde{A},W}$ are the same. The above properties is called the *gauge invariance* of the magnetic Schrödinger operator and it is in accordance with the fact that the physically relevant quantity is the magnetic field B and not its vector potential A (cf. Ref. 6).

Motivated by the theory of superconductivity, a lot of papers are devoted to the analysis of the spectrum of $L^{\hbar}_{A,W}$ in a semiclassical regime, namely as, $\hbar\to 0$. We quote in particular the works by Bernoff-Stenberg,[7] Del Pino–Felmer–Stenberg,[16] Lu–Pan,[7,26] devoted to the analysis, in a semiclassical regime, of the lowest eigenvalue of the magnetic Schrödinger operator. Finally we mention a recent paper by Helffer and Morame,[20] concerning the localization of the ground state in the case of a constant magnetic field.

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In the present work we study, in a semiclassical regime, a nonlinear Schrödinger equation with an additional cubic term, which arises in many fields of physics, in a particular condensed matter physics and nonlinear optics (see Ref. 33). More precisely, we are looking for stationary states to the evolution equation

$$i\hbar \frac{\partial \psi}{\partial t} = L^{\hbar}_{A,W}\psi - |\psi|^2\psi \quad \text{in } \mathbb{R}^+ \times \mathbb{R}^3, \tag{2}$$

as $\hbar \to 0$. The *ansatz* that the solution $\psi(x,t)$ to (2) is a standing wave of the form

$$\psi(t,x) = e^{-iE\hbar^{-1}t}u(x),$$

with $E \in \mathbb{R}$ and u: $\mathbb{R}^3 \to \mathbb{C}$, leads us to solve the semilinear elliptic equation

$$L^{\hbar}_{A,W}u = Eu + |u|^2u \quad \text{in } \mathbb{R}^3. \tag{3}$$

In the work we consider an electric potential $W(x)$ which is bounded from below on $\mathbb{R}^3$, and we choose E such that $V(x) = W(x) - E$ is strictly positive. Hence Eq. (3) becomes

$$L^{\hbar}_{A,V}u = |u|^2u \quad \text{in } \mathbb{R}^3, \tag{4}$$

where V is a strictly positive potential.

While there is an extensive literature dealing with (4) in the case $A=0$ (see Refs. 3, 4, 9, 10, 13, 15, 14, 19, 24, 27, and 29), there are few papers concerning the nonlinear Schrödinger equation with magnetic fields.

To our knowledge, the first paper, in which semilinear Schrödinger equation (4) with external magnetic field is considered, is by Esteban and Lions.[18] The authors proved the existence of standing wave solutions to (4), by a constrained minimization in the case $V(x)=1$ on $\mathbb{R}^3$ and $\hbar > 0$ is fixed. Concentration and compactness arguments are applied to solve such minimization problems for special classes of magnetic fields.

Afterward, in Ref. 21, Kurata has proved the existence of least energy solutions to (4) for any fixed $\hbar > 0$, under some assumptions linking the magnetic field B and the electric potential V (see also Ref. 32).

A first multiplicity result for standing wave solutions to (4), as $\hbar \to 0$, has been proved by Cingolani in Ref. 8, using topological arguments that allow to relate the number of standing wave solutions to (4) to the *topology* of the set of global minima of V. This result covers the case of magnetic potentials having polynomial growths, having special physical interest, but the used approach works only near global minima of V.

In a recent paper,[11] the more general case, in which the electric potential V has a manifold M of stationary points, not necessarily global minima, has been considered. For bounded electric and magnetic potentials, it has been proved a multiplicity result of semiclassical standing waves of (4), following the new perturbation approach contained in the paper[4] by Ambrosetti, Malchiodi, and Secchi (see also Refs. 2 and 3). Precisely, by means of a finite dimensional reduction, the complex-valued solutions to (4) are found near least energy solutions of the complex-valued limiting equation

$$\left(\frac{\nabla}{i} - A(\hbar\xi)\right)^2 u + u + V(\hbar\xi)u = |u|^2u \quad \text{in } \mathbb{R}^3, \tag{5}$$

where $\hbar\xi$ belongs to a neighborhood of M. We remark that in Ref. 11 the boundedness of the scalar and magnetic potentials on $\mathbb{R}^3$ is a crucial assumption to guarantee that the variational framework, in which problem (4) is set up, becomes *equivalent* to the space $H^1(\mathbb{R}^3,\mathbb{C})$, which is the variational setting of the limiting problem (5), independently of the vector potential A.

Concerning other papers on this topic, we mention a recent work[31] by Secchi and Squassina in which the authors have established necessary conditions for a sequence of standing wave solutions to (4) to concentrate, in different senses, around a given point. Finally we quote the paper by Arioli and Szulkin[5] where existence of infinitely many solutions of (5) is proved assuming that V and B are periodic and $\hbar$ fixed.

In the present work we are concerned with the study of standing wave solutions to (4) in a semiclassical regime, in the presence of a magnetic field, having source in a vector potential A possibly unbounded on $\mathbb{R}^3$. This is a relevant case in physics, since constant magnetic fields B lead to vector potentials A, having polynomial growths on $\mathbb{R}^3$. For instance, if B is the constant magnetic field $(0,0,b)$, then a suitable vector field is given by $A(x)=(b/2)(-x_2,x_1,0)$. In physical literature the potential A corresponds to the so-called *Lorentz gauge* (see Ref. 18).

In Main Theorem, which is the main result, we prove that for each topologically nontrivial critical point x_0 of the scalar potential V, there exists a standing wave solution $\psi_\hbar$ of (2) whose modulus concentrates at x_0 for $\hbar$ small. The magnetic field only influences the phase factor of the standing wave as $\hbar$ is small.

The used approach is variational and is based on a penalization procedure, introduced by Del Pino and Felmer in Ref. 13 for studying nonlinear Schrödinger equations with $A(x)=0$ in the semiclassical limit.

We point out that in the presence of a magnetic field new difficulties arise in order to carry out a penalization procedure as in Ref. 13. First, the problem is complex valued (unless $A\equiv 0$) and the penalization acts only on the modulus of the functions. Moreover differently from Refs. 11 and 13, if A is an unbounded function on $\mathbb{R}^3$, there is no kind of relationship between the variational setting $H^\hbar_{A,V}$ associated to problem (4), and the limit space $H^1(\mathbb{R}^3,\mathbb{C})$ as $\hbar\to 0$ (see Remark 3.1). Kato's inequality for magnetic fields and delicate subsolution estimates will provide the tools to extend the results in Ref. 13 for nonlinear Schrödinger equations in presence of an external magnetic field.

We use the following notations:

(1) The complex conjugate of any number $z\in\mathbb{C}$ will be denoted by $\bar{z}$.
(2) The real part of a number $z\in\mathbb{C}$ will be denoted by $\operatorname{Re} z$.
(3) The ordinary inner product between two vectors $a,b\in\mathbb{R}^3$ will be denoted by $a\cdot b$.
(4) From time to time, when no confusion can arise, we omit the symbol dx in integrals over $\mathbb{R}^3$.
(5) The letter C denotes a generic positive constant, which may vary inside a chain of inequalities.
(6) We use the Landau symbols. For example, $O(\varepsilon)$ is a generic function such that $\limsup_{\varepsilon\to 0}[O(\varepsilon)/\varepsilon]<\infty$, and $o(\varepsilon)$ is a function such that $\lim_{\varepsilon\to 0}[o(\varepsilon)/\varepsilon]=0$.

II. STATEMENT OF THE MAIN RESULT

In the work we consider, more generally, with the semilinear elliptic equation

$$\left(\frac{\hbar}{i}\nabla - A(x)\right)^2 u + V(x)u = f(|u|^2)u \quad \text{in } \mathbb{R}^3, \tag{6}$$

where $\hbar$ is regarded as a small parameter and $f:[0,+\infty[\to\mathbb{R}$ satisfies the following assumptions.

(f1) f is of class C^1 increasing, $f(0)=0$ and

$$\lim_{s\to\infty}\frac{f(s)}{s^{\frac{p-1}{2}}}=0 \quad \text{and } 0<\vartheta F(s)\leqslant f(s)s$$

for some $p\in(1,5)$ and $\vartheta>2$, where $F(s)=\frac{1}{2}\int_0^s f(t)\mathrm{d}t$, for $s\in\mathbb{R}^+$.

(f2) For each $a>0$, the limiting functional $I_a:H^1(\mathbb{R}^3,\mathbb{R})\to\mathbb{R}$, defined as

$$I_a(v) = \frac{1}{2}\int_{\mathbb{R}^3} [|\nabla v|^2 + a|v|^2] - \int_{\mathbb{R}^3} F(|v|^2), \tag{7}$$

possesses a unique critical point, whose critical value is denoted by b^a.

Throughout the paper we also make the following mild assumptions on the vector and scalar potentials.

(A1) A: $\mathbb{R}^3 \to \mathbb{R}^3$ is a C^1-vector field such that, for some positive constants C and γ,

$$|J_A(x)| \leqslant Ce^{\gamma|x|}, \tag{8}$$

where $J_A(x)$ denotes the Jacobian matrix of A at x.

(V1) $V: \mathbb{R}^3 \to \mathbb{R}$ is a positive C^1 function such that $\inf_{x\in\mathbb{R}^3} V(x) = V_0 > 0$ and for some positive constants C_1 and γ_1,

$$|\nabla V(x)| \leqslant C_1 e^{\gamma_1|x|}. \tag{9}$$

(V2) There is an open, bounded set $\Lambda \subset \mathbb{R}^3$ with smooth boundary and there exist closed subsets B, B_0 of Λ such that B is connected and $B_0 \subset B$. Let Γ be the family of all continuous functions ϕ: $B \to \Lambda$ with the property that $\phi(x) = x$ whenever $x \in B_0$. Define

$$c = \inf_{\phi\in\Gamma} \max_{x\in B} V(\phi(x)). \tag{10}$$

Moreover we assume that $\sup_{x\in B_0} V(x) < c$ and for all $\phi \in \Gamma$,

$$c \leqslant \inf_{x\in B} V(\phi(x)).$$

(V3) For all $x \in \partial\Lambda$ such that $V(x) = c$, there holds $\partial_\tau V(x) \neq 0$, where ∂_τ stands for the tangential derivative.

We notice that assumptions (V2) and (V3) express a local linking for V in Λ (see for instance Ref. 14) and guarantee the existence of a critical point for V inside Λ at level c. Particular cases of local linking of V in Λ are local maxima, local minima or saddle points for V inside Λ.

We can state the main result of this work, which is going to be proved in the last section.

Main Theorem: *Assume* (f1) and (f2), (A1), (V1)–(V3). *Then there is a number* $\hbar_0 > 0$ *such that for all* $0 < \hbar < \hbar_0$, *there exists a solution* $u_\hbar$ *to Eq.* (6) *such that*

$$\int_{\mathbb{R}^3} \left| \left(\frac{\hbar}{i}\nabla - A(x)\right) u_\hbar \right|^2 \mathrm{d}x + \int_{\mathbb{R}^3} V(x)|u_\hbar|^2\, \mathrm{d}x < +\infty. \tag{11}$$

Furthermore $u_\hbar \in C^{2,\alpha}_{\mathrm{loc}}(\mathbb{R}^3)$, *with* $\alpha \in (0,1)$.

We remark that assumption (f1) is clearly satisfied if the nonlinear term in (6) is homogeneous, namely $f(t) = |t|^{(p-2)/2}$. In this case, assumption (f2) is also satisfied by the uniqueness results in Ref. 22. By Main Theorem, we deduce the following corollary.

Corollary 2.1: Assume (A1), (V1)–(V3). *Then there is a number* $\hbar_0 > 0$ *such that for all* $0 < \hbar < \hbar_0$, *there exists a solution* $u_\hbar$ *to Eq.* (4) *such that* (11) *holds. Furthermore* $u_\hbar \in C^{2,\alpha}_{\mathrm{loc}}(\mathbb{R}^3)$, *with* $\alpha \in (0,1)$.

We remark that in Main Theorem we deal with a nonlinear Schrödinger equation in $\mathbb{R}^3$ as this is the main relevant case in quantum mechanics. Actually, the result of Main Theorem also holds for nonlinear Schrödinger equations in $\mathbb{R}^N$, assuming the following.

(f2′) The nonlinearity f is of class C^1, increasing, $f(0) = 0$ and

$$\lim_{s\to\infty}\frac{f(s)}{s^{\frac{p-1}{2}}}=0 \quad \text{and } 0<\vartheta F(s)\leqslant f(s)s$$

for some $p>1$ if $N=1, 2$ and $p\in\left[1,(N+2)/(N-2)\right]$ if $N\geqslant 3$ and $\vartheta>2$, where $F(s)=\frac{1}{2}\int_0^s f(t)\mathrm{d}t$, for $s\in\mathbb{R}^+$.

III. MAGNETIC FIELDS: THE SPACE H_A

In this section we recall some classical results on Schrödinger operators with magnetic field, which are useful in the proof of Main Theorem.

We consider the space $H_A(\mathbb{R}^3,\mathbb{C})$ consisting of all the function $u\in L^2(\mathbb{R}^3,\mathbb{C})$ with $(\partial_j+iA_j)u\in L^2(\mathbb{R}^3,\mathbb{C})$ for any $j=1,2,3$ endowed with the norm

$$\|u\|_{H_A}^2=\int_{\mathbb{R}^3}|(\nabla+iA)u|^2\,\mathrm{d}x+\int_{\mathbb{R}^3}|u|^2\,\mathrm{d}x.$$

Remark 3.1: We do not assume that ∇u *or* Au *are separately in* $L^2(\mathbb{R}^3,\mathbb{C})$*. Therefore, in general, there is no relationship between the spaces* $H_A(\mathbb{R}^3,\mathbb{C})$ *and* $H^1(\mathbb{R}^3,\mathbb{C})$*, namely* $H_A(\mathbb{R}^3,\mathbb{C})\not\subset H^1(\mathbb{R}^3,\mathbb{C})$ *or* $H^1(\mathbb{R}^3,\mathbb{C})\not\subset H_A^{\hbar}(\mathbb{R}^3,\mathbb{C})$ *(see Ref. 18).*

Theorem 3.2: *Let* A: $\mathbb{R}^3\to\mathbb{R}^3$ *be in* $L^2_{\mathrm{loc}}(\mathbb{R}^3)$ *and let* $u\in H_A^1(\mathbb{R}^3,\mathbb{C})$*. Then* $|u|\in H^1(\mathbb{R}^3,\mathbb{R})$ *and the diamagnetic inequality*

$$|\nabla|u|(x)|\leqslant|(\nabla+iA)u(x)| \tag{12}$$

holds for almost every $x\in\mathbb{R}^3$.

By the diamagnetic inequality, the following result follows (see Ref. 18).

Theorem 3.3: *The space* $C_0^\infty(\mathbb{R}^3,\mathbb{C})$ *is dense in* $H_A^1(\mathbb{R}^3,\mathbb{C})$.

Furthermore we recall the following Kato's inequality (see Ref. 30).

Theorem 3.4: *Let* $u\in L^1_{\mathrm{loc}}(\mathbb{R}^3,\mathbb{C})$ *with* $\nabla u\in L^1_{\mathrm{loc}}(\mathbb{R}^3,\mathbb{C})$*. Define*

$$(\mathrm{sign}\,u)(x)=\begin{cases}\dfrac{\overline{u(x)}}{|u(x)|} & if\ u(x)\neq 0,\\ 0 & if\ u(x)=0,\end{cases} \tag{13}$$

we have that $\mathrm{sign}\,u\in L^\infty(\mathbb{R}^3)$ *and* $(\mathrm{sign}\,u)\nabla u(x)$ *is locally* L^1 *and hence a distribution. Moreover we have*

$$\Delta|u|\geqslant\mathrm{Re}[(\mathrm{sign}\,u)\Delta u].$$

We furthermore recall the application of Kato's inequality to the Schrödinger operator with magnetic field (see Ref. 30).

Theorem 3.5: *Let* A: $\mathbb{R}^3\to\mathbb{R}^3$ *be a* C^1 *real vector valued function. Let* $D_k u=(1/i)(\partial u/\partial x_k)-A_k u$*, for any* $k=1,2,3$ *and* $D_A^2=\Sigma_{k=1}^3 D_k^2$*. Then for any* $u\in L^1_{\mathrm{loc}}(\mathbb{R}^3,\mathbb{C})$ *and* $D^2u\in L^2_{\mathrm{loc}}(\mathbb{R}^3,\mathbb{C})$ *we have*

$$\Delta|u|\geqslant-\mathrm{Re}[(\mathrm{sign}\,u)D_A^2u].$$

Throughout the paper, we set $\hbar=\varepsilon$ and denote by D_ε and D^ε for each $\varepsilon>0$ the (formal) differential operators

$$D_\varepsilon=\frac{\varepsilon}{i}\nabla-A(x), \tag{14}$$

$$D^{\varepsilon} = \frac{\nabla}{i} - A(\varepsilon x). \tag{15}$$

As in Sec. II, we introduce the real Hilbert space $H^{\varepsilon}_{A,V}$ as the completion of $C_0^{\infty}(\mathbb{R}^3,\mathbb{C})$ with respect to the inner product

$$\langle u,v\rangle_{H^{\varepsilon}_{A,V}} = \mathrm{Re}\int_{\mathbb{R}^3} D_{\varepsilon}u \cdot \overline{D_{\varepsilon}v}\, \mathrm{d}x + \mathrm{Re}\int_{\mathbb{R}^3} V(x)u\bar{v}\, \mathrm{d}x. \tag{16}$$

As remarked above, this space has in general no relationship with $H^1(\mathbb{R}^3,\mathbb{C})$. Anyway, by Theorem 3.2, we have the following *diamagnetic inequality*:

$$\int_{\mathbb{R}^3} \varepsilon^2 |\nabla |u||^2\, \mathrm{d}x \leq \int_{\mathbb{R}^3} |D_{\varepsilon}u|^2\, \mathrm{d}x, \quad \text{for every } u \in H^{\varepsilon}_{A,V}. \tag{17}$$

It is easy to check that, under our assumptions, the functional

$$\mathcal{F}_{\varepsilon}(u) = \frac{1}{2}\int_{\mathbb{R}^3} |D_{\varepsilon}u|^2\, \mathrm{d}x + \int_{\mathbb{R}^3} V(x)|u|^2\, \mathrm{d}x - \int_{\mathbb{R}^3} F(|u|^2)\mathrm{d}x \tag{18}$$

is of class C^2, so that solutions to (6) correspond to critical points of $\mathcal{F}_{\varepsilon}$.

IV. A PENALIZATION ACTING ON THE MODULUS

In this section we perform a penalization of the Euler functional $\mathcal{F}_{\varepsilon}$, inspired by Refs. 13 and 14. To this order, we begin to assume, without loss of generality, that the infimum of V in $\overline{\Lambda}$ is very close to c. Let $\delta > 0$ be a small but fixed number, we can assume that $\Lambda = \{x \mid V(x) > c - \delta\}$ and

$$B \subset \{x \in \Lambda | V(x) \geq c(\varepsilon)\}, \quad B_0 \subset \{x \in \Lambda | V(x) = c(\varepsilon)\},$$

where $c-\delta < c(\varepsilon) < c$, $\lim_{\varepsilon\to 0} c(\varepsilon) = c - \delta$ and $\mathrm{dist}(B_0, \partial\Lambda) = \sqrt{\varepsilon}$. In fact as in Ref. 14, we can redefine $\Lambda_{\delta} = \Lambda \cap \{x \mid V(x) > c - \delta\}$ and

$$B^{\delta,\varepsilon} = B \cap \{x | V(x) \geq c(\varepsilon)\}, \tag{19}$$

$$B_0^{\delta,\varepsilon} = B_0 \cap \{x | V(x) = c(\varepsilon)\}, \tag{20}$$

where

$$c(\varepsilon) = \inf\{\xi | \mathrm{dist}(\{x \in \Lambda | V(x) = \xi\}, \Lambda_{\delta}) \geq \sqrt{\varepsilon}\}$$

without affecting condition (V3) in the definition of linking. We notice that the set $B_0^{\delta,\varepsilon}$ is not empty as B is connected. Then if ϕ: $B^{\delta,\varepsilon} \to \Lambda_{\delta}$ is a continuous map with $\phi(x) = x$ for every $x \in B_0^{\delta,\varepsilon}$, we can define its extension $\tilde{\phi}$ as the identity on $B \backslash B^{\delta,\varepsilon}$. Thus $\tilde{\phi}$: $B \to \Lambda$ and $\sup_{x\in B} V(\tilde{\phi}) = \sup_{x\in B^{\delta,\varepsilon}} V(\tilde{\phi}) \geq c$.

We consider a modification of the nonlinear term in (6), that will prevent concentration outside Λ. We remark that differently from Ref. 13, as our problem is complex valued, the penalization affects only the modulus of the functions.

Let ϑ be the number defined in (f1) and choose $k > 0$ such that $k > \vartheta/(\vartheta - 2)$. Since f is increasing, we can fix a number $a > 0$ with $f(a) = V_0/k$. Set

$$\tilde{f}(s) = \begin{cases} f(s) & \text{if } s \leq a, \\ V_0/k & \text{if } s > a, \end{cases} \tag{21}$$

we define g: $\mathbb{R}^3 \times \mathbb{R}^+ \to \mathbb{R}$ by

$$g(x,s) = \chi_\Lambda(x)f(s) + (1-\chi_\Lambda(x))\tilde{f}(s), \tag{22}$$

where χ_Λ is the characteristic function of the set Λ and we consider the modified equation

$$\left(\frac{\varepsilon}{i}\nabla - A(x)\right)^2 u + V(x)u = g(x,|u|^2)u \quad \text{in } \mathbb{R}^3. \tag{23}$$

Weak solutions of Eq. (23) correspond to critical points of the C^1 functional J_ε: $H^\varepsilon_{A,V} \to \mathbb{R}$,

$$J_\varepsilon(u) = \frac{1}{2}\int_{\mathbb{R}^3} |D_\varepsilon u|^2 + V(x)|u|^2 - \int_{\mathbb{R}^3} G(x,|u|^2)\mathrm{d}x, \tag{24}$$

where $G(x,s) = \frac{1}{2}\int_0^s g(x,t)\mathrm{d}t$.

For the sake of convenience, we highlight some obvious properties of g, which follow directly from (f1) and (f2).

(g1) $\lim_{s\to 0^+} g(x,s)=0$, uniformly with respect to $x\in\mathbb{R}^3$.

(g2) There exist a bounded subset K of $\mathbb{R}^3$ and a number $\vartheta > 2$ such that

$$0 < \vartheta G(x,s) \leqslant g(x,s)$$

for all $x\in K$.

(g3) For all $s\geqslant 0$, $x\notin K$,

$$0 \leqslant 2G(x,s) \leqslant g(x,s) \leqslant \frac{1}{k}V(x)$$

with a constant $k > \vartheta/(\vartheta-2)$.

We begin to show that the penalized functional J_ε satisfies the Palais–Smale condition. This may not be true for the functional $\mathcal{F}_\varepsilon$.

Lemma 4.1: For any $\varepsilon>0$ fixed, the penalized functional J_ε satisfies the Palais–Smale condition at all positive levels.

Proof: Let $\varepsilon>0$ be fixed. Let $\{u_n\}$ be a sequence in $H^\varepsilon_{A,V}$ such that $\{J_\varepsilon(u_n)\}$ is bounded and $J'_\varepsilon(u_n)\to 0$. First we prove that $\{u_n\}$ is bounded. By (g3), it follows that

$$\frac{1}{2}\int_K g(x,|u_n|^2)|u_n|^2 + o(\|u_n\|) \leqslant \frac{1}{2}\int_{\mathbb{R}^3} |D_\varepsilon u_n|^2 + V|u_n|^2 \leqslant \int_K G(x,|u_n|^2) + \frac{1}{2k}\int_{\mathbb{R}^3\setminus K} V|u_n|^2 + O(1),$$

where $\|\cdot\|$ denotes the norm in $H^\varepsilon_{A,V}$ induced by the scalar product in (16). Thus the above inequality and (g2) imply

$$\left(\frac{\vartheta}{2}-1\right)\int_{\mathbb{R}^3} |D_\varepsilon u_n|^2 + V|u_n|^2 \leqslant \frac{\vartheta}{2k}\int_{\mathbb{R}^3\setminus K} V|u_n|^2 + o(\|u_n\|) + O(1).$$

In particular, it follows that $\{u_n\}$ is bounded in $H^\varepsilon_{A,V}$. We choose a subsequence, still denoted by $\{u_n\}$ for simplicity, that converges weakly to some u in $H^\varepsilon_{A,V}$. we claim that $u_n\to u$ strongly in $H^\varepsilon_{A,V}$. To this aim, it suffices to show that for any given $\delta>0$ there exists $R>0$ such that

$$\limsup_{n\to\infty}\int_{|x|>R} (|D_\varepsilon u_n|^2 + V(x)|u_n|^2)\mathrm{d}x < \delta.$$

Without loss of generality, we can take R so large that $K\subset B_{R/2}$. Fix a smooth cutoff function η_R such that $\eta_R=0$ on $B_{R/2}$, $\eta_R=1$ outside $B_{R/2}$, $0\leqslant\eta_R\leqslant 1$ and $|\nabla\eta_R|\leqslant c/R$ for some constant $c>0$. Since $\{u_n\}$ is a bounded Palais–Smale sequence, we have

$$J'(u_n)[\eta_R u_n] = o(1),$$

so that

$$\int_{\mathbb{R}^3} (|D_\varepsilon u_n|^2 + V|u_n|^2)\eta_R + \mathrm{Re}\int_{\mathbb{R}^3} u_n D_\varepsilon u_n \cdot \overline{D_\varepsilon \eta_R} = \mathrm{Re}\int_{\mathbb{R}^3} g(x,|u_n|^2)\eta_R |u_n|^2 + o(1)$$
$$\leqslant \frac{1}{k}\int_{\mathbb{R}^3} V|u_n|^2 \eta_R + o(1).$$

We conclude that

$$\int_{|x|>R} |D_\varepsilon u_n|^2 + V(x)|u_n|^2 \leqslant \frac{C}{R}\|u_n\|_{L^2}\|D_\varepsilon u_n\|_{L^2} + o(1),$$

which clearly proves the claim. □

V. THE MINIMAX SCHEME

By assumption (f2) the limiting functional I_a defined in (7) has a unique critical value, which we can characterize as

$$b^a = \inf_{v\in H^1(\mathbb{R}^3)\setminus\{0\}} \sup_{t>0} I_a(tv). \tag{25}$$

It can be shown that the map $a \mapsto b^a$, with $a>0$, is strictly increasing and continuous. Associated to the critical value b^a there exists a radially symmetric solution $\omega_a \in H^1(\mathbb{R}^3,\mathbb{R})$ of the scalar equation

$$\Delta\omega - a\omega + f(|\omega|^2)\omega = 0. \tag{26}$$

Fix a small number $\delta_0 > 0$. For each $y \in \mathbb{R}^3$ with $\mathrm{dist}(y,\partial\Lambda) > \delta_0$ we denote by w_ε^y the function in $H_{A,V}^\varepsilon$ given by

$$w_\varepsilon^y(x) = e^{iA(y)[(y-x)/\varepsilon]}\eta(|x-y|/\delta_0)\omega_{V(y)}\left(\frac{y-x}{\varepsilon}\right), \tag{27}$$

where η is a smooth cutoff function that equals 1 on (0, 1) and 0 on $(2,+\infty)$.

Define now the class Γ_ε of all continuous maps ϕ: $B^\varepsilon \to \mathcal{M}_\varepsilon$ such that

$$\phi(y) = t(\varepsilon,y)w_\varepsilon^y \quad \text{for all } y \in B_0^\varepsilon, \tag{28}$$

where

$$\mathcal{M}_\varepsilon = \left\{ u \in H_{A,V}^\varepsilon \setminus \{0\} \middle| \int_{\mathbb{R}^3} |D_\varepsilon u|^2 + V(x)|u|^2 = \int_{\mathbb{R}^3} g(x,|u|^2)|u|^2 \right\}$$

is the Nehari manifold associated to the polarized functional J_ε and $t(\varepsilon,y)$ is the unique positive number such that $t(\varepsilon,y)w_\varepsilon^y \in \mathcal{M}_\varepsilon$. We define a minimax value as follows:

$$\gamma_\varepsilon = \inf_{\phi\in\Gamma_\varepsilon} \sup_{y\in B^\varepsilon} J_\varepsilon(\phi(y)). \tag{29}$$

By slightly deforming w_ε^y and recalling the definitions of B^ε and B_0^ε, one can show that

$$b^c \geqslant \limsup_{\varepsilon\to 0} \varepsilon^{-3}\gamma_\varepsilon \geqslant \liminf_{\varepsilon\to 0} \varepsilon^{-3}\gamma_\varepsilon \geqslant b^{c-\delta}. \tag{30}$$

We show that the last inequality in (30) is strict. We begin to prove the following useful lemma, which generalizes Lemma 2.3 in Ref. 14 (see also Ref. 17) to the case of a complex valued equation.

Lemma 5.1: Let $v \in H^1(\mathbb{R}^3, \mathbb{C}) \cap C(\mathbb{R}^3)$ *be a weak solution of the equation*

$$\Delta v - V(\xi)v + \chi_{\{x_1<0\}} f(|v|^2)v + \chi_{\{x_1>0\}} \tilde{f}(|v|^2)v = 0, \tag{31}$$

where $\xi \in \mathbb{R}^3$. *Then* $|v| \leqslant \sqrt{a}$ *for any* $x_1 > 0$ *and* v *actually solves the equation*

$$\Delta v - V(\xi)v + f(|v|^2)v = 0.$$

Proof: We test Eq. (31) by $\overline{\partial v}/\partial x_1$ and we derive [x' stands for $=(x^2, x^3)$]

$$\int_{\mathbb{R}^2} \mathrm{d}x' \int_{-\infty}^{\infty} \frac{\partial}{\partial x_1}[|\nabla v|^2 + V(\xi)|v|^2]\mathrm{d}x_1 + \int_{\mathbb{R}^2} \Big[F(|v(0,x')|^2) - \tilde{F}(|v(0,x')|^2)\Big]\mathrm{d}x' = 0. \tag{32}$$

We notice that $F(s) \geqslant \tilde{F}(s)$ with inequality if $s \leqslant a$. Thus $|v(0,x')| \leqslant \sqrt{a}$. Finally we show that $|v(x_1,x')|^2 \leqslant \sqrt{a}$ for any $x_1 > 0$.

By Kato's inequality we derive that

$$\Delta|v| \geqslant V(\xi)|v| - \chi_{\{x_1<0\}} f(|v|^2)|v| - \chi_{\{x_1>0\}} \tilde{f}(|v|^2)|v|. \tag{33}$$

Now we can test (33) by $\phi = \chi_{\{x_1>0\}}(|v| - \sqrt{a})_+ \in H^1(\mathbb{R}^3, \mathbb{R})$ and we derive

$$\int_{\mathbb{R}^3} \chi_{\{x_1>0\}} |\nabla(|v| - \sqrt{a})_+|^2 + q(x)\chi_{\{x_1>0\}}(|v| - \sqrt{a})_+^2 + \sqrt{a}q(x)\chi_{\{x_1>0\}}(|v| - \sqrt{a})_+ \leqslant 0, \tag{34}$$

where

$$q(x) = V(\xi) - \tilde{f}(|v|^2)\chi_{\{x_1>0\}}.$$

For $s \geqslant a$, $\tilde{f}(s) = (V_0/k) < V(\xi)$, so that $q(x) > 0$ and all the terms in (34) are necessarily zero, and $\phi = \chi_{\{x_1>0\}}(|v| - \sqrt{a})_+ = 0$. We conclude that

$$|v(x_1,x')| \leqslant \sqrt{a} \quad \forall\, x_1 > 0, \ \ x' \in \mathbb{R}^2.$$

□

Lemma 5.2: There results

$$\liminf_{\varepsilon \to 0} \varepsilon^{-3}\gamma_\varepsilon > b^{c-\delta}. \tag{35}$$

Proof: We argue by contradiction, following arguments strictly related to Ref. 14, Lemma 1.1. If (35) is not true, then there exists a sequence $\varepsilon_n \to 0$ such that

$$\varepsilon_n^{-3}\gamma_{\varepsilon_n} \leqslant b^{c-\delta} + o(1).$$

Fix some $\phi_n \in \Gamma_{\varepsilon_n}$ with the property that

$$\varepsilon_n^{-3} \sup_{y \in B^{\varepsilon_n}} J_{\varepsilon_n}(\phi_n(y)) \leqslant b^{c-\delta} + o(1). \tag{36}$$

For the reader's convenience, we split the proof in several steps.

Step I: Setting $\Lambda_n = \{x \in \mathbb{R}^3 | \mathrm{dist}(x, \Lambda) < \sqrt{\varepsilon_n}\}$, we claim that

$$\lim_{n\to\infty} \varepsilon_n^{-3} \sup_{y\in B^{\varepsilon_n}} \int_{\mathbb{R}^3\setminus\Lambda_n} |\phi_n(x)|^2 = 0. \tag{37}$$

To prove this claim, we fix $y_n \in B^{\varepsilon_n}$ and simplify notation by introducing $u_n = \phi_n(y_n)$.

Since $u_n \in \mathcal{M}_{\varepsilon_n}$, we have

$$J_{\varepsilon_n}(u_n) \geqslant J_{\varepsilon_n}(tu_n)$$

for any $t > 0$. Let us set

$$E_n(v) = \frac{1}{2}\int_{\Lambda_n} |D_{\varepsilon_n} v|^2 + V(x)|v|^2 - \int_{\Lambda_n} G(x,|v|^2)\mathrm{d}x,$$

and choose numbers $t_n > 0$ with the property that

$$E_n(t_n u_n) = \max_{t>0} E_n(tu_n).$$

Now, from the properties of the penalization g, it follows that

$$\frac{V(x)}{2} - G(x,s) \geqslant \gamma \quad \text{for all } x \in \mathbb{R}^3 \setminus \Lambda \text{ and } s > 0.$$

This and Eq. (36) imply that

$$E_n(t_n u_n) + \gamma t_n^2 \int_{R^3\setminus\Lambda_n} |u_n|^2 \leqslant \varepsilon_n^3 (b^{c-\delta} + o(1))$$

with an error $o(1)$ uniform with respect to $\{y_n\}$. Furthermore, we claim that there exists $\sigma > 0$ such that

$$\inf_{n\geqslant 1} t_n \geqslant \sigma. \tag{38}$$

First we notice that from the relation $J_{\varepsilon_n}(u_n) \leqslant C\varepsilon_n^3$, the diamagnetic inequality and again the properties of g, the existence of a constant C_0, independent of $\{y_n\}$, such that

$$\int_{\mathbb{R}^3} |D_{\varepsilon_n} u_n|^2 + |u_n|^2 \leqslant C_0 \varepsilon_n^3 \tag{39}$$

follows easily. Set now $v_n(x) = t_n u_n(\varepsilon_n x)$ and $\tilde{\Lambda}_n = \varepsilon_n^{-1}\Lambda_n$. The definition of t_n implies

$$\int_{\tilde{\Lambda}_n} |D^{\varepsilon_n} v_n|^2 + V(\varepsilon_n x)|v_n|^2 = \int_{\tilde{\Lambda}_n} g(\varepsilon_n x, |v_n|^2)|v_n|^2 \,\mathrm{d}x \leqslant \int_{\tilde{\Lambda}_n} C_0 |v_n|^{p+1} + \rho |v_n|^2, \tag{40}$$

where $\rho > 0$ can be fixed as small as we please. We can deduce from the Sobolev embedding theorem as stated in Ref. 1, Lemma 5.10 that there exists a constant $\overline{C} > 0$, independent of n, such that

$$\int_{\tilde{\Lambda}_n} |v_n|^{p+1} \leqslant \overline{C}\left(\int_{\tilde{\Lambda}_n} |\nabla |v_n||^2 + |v_n|^2\right)^{(p+1)/2} \leqslant \overline{C}\left(\int_{\tilde{\Lambda}_n} |D^{\varepsilon_n} v_n|^2 + |v_n|^2\right)^{(p+1)/2}. \tag{41}$$

By combining (41) with Eq. (40), we see that

$$\int_{\tilde{\Lambda}_n} |v_n|^{p+1} \geqslant \sigma > 0,$$

and in particular $\int_{\tilde{\Lambda}_n} |D^{\varepsilon_n} v_n|^2 + |v_n|^2 \geqslant \sigma > 0$ for a suitable $\sigma > 0$ independent of n, and so

$$t_n^2 \int_{\Lambda_n} (|D_{\varepsilon_n} u_n|^2 + |u_n|^2) \geqslant \sigma \varepsilon_n^3.$$

This proves (38).

Observe now that, from the definition of t_n and from the diamagnetic inequality, we get

$$E_n(t_n u_n) \geqslant \inf_{u \in H^1(\Lambda_n)} \sup_{t>0} E_n(tu) \equiv b_n.$$

If we prove that

$$\lim_{n\to\infty} \varepsilon_n^{-3} b_n = b^{c-\delta}, \tag{42}$$

then Eq. (37) follows from our previous arguments.

Step II: We prove that identity (42) holds.

We follow Ref. 15, with minor changes. By a deformation argument, it is easy to see that

$$b_n \leqslant (b^{c-\delta} + o(1))\varepsilon_n^3. \tag{43}$$

We prove the opposite inequality. Since the functional E_n satisfies (PS), by standard Critical Point Theory, the number b_n is a critical value for E_n. Let $w_n \in H^1(\Lambda_n)$ be an associated critical point. As such, it satisfies the equation

$$\begin{cases} \left(\dfrac{\varepsilon_n}{i}\nabla - A\right)^2 w_n + V w_n = g(x,|w_n|^2) w_n \quad \text{in } \Lambda_n, \\ \dfrac{\partial w_n}{\partial \nu} = 0 \quad \text{on } \partial\Lambda_n. \end{cases}$$

In particular, by Kato's inequality, $|w_n|$ solves the differential inequality

$$\begin{cases} \varepsilon_n^2 \Delta |w_n| - V(x)|w_n| + g(x,|w_n|^2)|w_n|^2 \geqslant 0 \quad \text{in } \Lambda_n, \\ \dfrac{\partial w_n}{\partial \nu} = 0 \quad \text{on } \partial\Lambda_n. \end{cases} \tag{44}$$

By the maximum principle, $|w_n|$ cannot attain a local maximum inside $\overline{\Lambda_n} \setminus \Lambda$, thanks to the Neumann boundary condition in (44). If x_n is a maximum of $|w_n|$, then necessarily $x_n \in \overline{\Lambda}$. Moreover, $\inf_n \max_{\Lambda_n} |w_n| > 0$. Assume, without loss of generality, that $x_n \to x^* \in \overline{\Lambda}$. Scaling w_n to a map on $\Omega_n \equiv \varepsilon_n^{-1}(\Lambda_n - x_n)$ defined by $v_n(x) = w_n(x_n + \varepsilon_n x)$, we have that v_n satisfies the equation

$$\begin{aligned} &-\Delta v_n - \frac{2}{i} A(x_n + \varepsilon_n x) \cdot \nabla v_n + |A(x_n + \varepsilon_n x)|^2 v_n - \frac{\varepsilon_n}{i} \operatorname{div} A(x_n + \varepsilon_n x) v_n \\ &\quad + V(x_n + \varepsilon_n x) v_n = g(x_n + \varepsilon_n x, |v_n|^2) v_n \quad \text{in } \Omega_n \end{aligned} \tag{45}$$

with Neumann boundary condition, and, again by Kato's inequality,

$$\Delta |v_n| \geqslant V(x_n + \varepsilon_n x)|v_n| - g(x_n + \varepsilon_n x, |v_n|^2)|v_n| \quad \text{in } \Omega_n.$$

From (43) we deduce that

$$\sup_{n\geq 1}\int_{\Omega_n}[|D^{\varepsilon_n}v_n|^2\,\mathrm{d}x + V_n|v_n|^2]\mathrm{d}x < +\infty,$$

where $A_n(x)=A(x_n+\varepsilon_n x)$ and $V_n(x)=V(x_n+\varepsilon_n x)$. Take an arbitrary open set Ω, relatively compact in $\mathbb{R}^3$. Since we may assume that $\Omega\subset\Omega_n$ for all n sufficiently large, Eq. (43) and the diamagnetic inequality (12), entail that the sequence $\{|v_n|\}$ is bounded in $H^1(\Omega,\mathbb{R})$ and, up to subsequences, converges weakly in $H^1(\Omega,\mathbb{R})$ and strongly in $L^q(\Omega,\mathbb{R})$ with $q<5$ to some $v^*\in H^1(\Omega,\mathbb{R})$. Moreover $\{v_n\}$ is a bounded sequence in $H^1(\Omega,\mathbb{C})$. Since Ω is arbitrary, the limit v^* can be extended to a function defined on $\mathbb{R}^3$. Thus by applying the subsolution estimates (see Theorems 13.1, 14.1 in Ref. 23) we infer that the sequence $\{v_n\}$ is bounded in $L^\infty(\Omega)$. By Schauder estimates, the sequence $\{v_n\}$ is bounded in $C^{2,\alpha}(K)$ for some $\alpha\in(0,1)$ and thus, up to subsequences, v_n converges to v in $C^2_{\mathrm{loc}}(\mathbb{R}^3)$ and also weakly in $L^q(\mathbb{R}^3,\mathbb{R})$ with $q<5$. It follows that $|v|=v^*\in H^1(\mathbb{R}^3,\mathbb{R})$ and $v\neq 0$ as $\inf_n|v_n(0)|\geq b>0$. Therefore,

$$-\Delta v - \frac{2}{i}A(x^*)\cdot\nabla v + |A(x^*)|^2 v + V(x^*)v = \bar{g}(x,|v|^2)v \quad \text{in } \mathbb{R}^3, \tag{46}$$

in the sense of distributions, where $\bar{g}(x,s)=\chi(x)f(s)+(1-\chi(x))\tilde{f}(s)$ and χ is the weak * limit of the sequence $\{\chi_\Lambda(x_n+\varepsilon_n)\}_{n\geq 1}$ in $L^\infty(\mathbb{R}^3)$. Since $|v|\in H^1(\mathbb{R}^3,\mathbb{R})$ and by definition of $\bar{g}$, we deduce that $\int_{\mathbb{R}^3}\bar{g}(x,|v|^2)|v|^2\,\mathrm{d}x$ is finite and by (46) we have $v\in H^1(\mathbb{R}^3,\mathbb{C})$ and thus v solves (46) in weak sense. By performing a rotation, Lemma 5.1 can be applied to prove that the function $v(x)=e^{-\mathrm{i}A(x^*)\cdot x}\tilde{v}(x)$ satisfies

$$\Delta v - V(x^*)v + f(|v|^2)v = 0.$$

We must have

$$\liminf_{n\to\infty}\varepsilon_n^{-3}E_n(w_n) = \liminf_{n\to\infty}E_n(v_n) \geq I_{V(x^*)}(v).$$

Indeed, as v_n converges to v in $C^2_{\mathrm{loc}}(\mathbb{R}^3)$, we derive that

$$\lim_{n\to+\infty}\int_{B_R}\sigma_n = \frac{1}{2}\int_{B_R}\left|\left(\frac{\nabla}{i}-A(x^*)\right)v\right|^2 + \frac{V(x^*)}{2}\int_{B_R}|v|^2 - \int_{B_R}\bar{G}(x,|v|^2), \tag{47}$$

where

$$\sigma_n(x) = \frac{1}{2}\left[\left|\left(\frac{\nabla}{i}-A_n(x)\right)v_n\right|^2 + V_n(x)|v_n|^2\right] - \bar{G}(x,|v_n|^2).$$

Since $v\in H^1(\mathbb{R}^N,\mathbb{C})$, we have that for each $\delta>0$ there exists $R>0$ so large that

$$\lim_{n\to+\infty}\int_{B_R}\sigma_n \geq I_{V(x^*)}(v) - \delta.$$

To complete the proof, we need to show that

$$\liminf_{n\to\infty}\int_{\varepsilon_n^{-1}(\Lambda_n-x_n)\setminus B_R}\sigma_n(x)\mathrm{d}x \geq -\delta \tag{48}$$

for R sufficiently large. Choose a smooth cutoff function η such that $\eta=0$ on B_{R-1}, $\eta=1$ on $\mathbb{R}^3\setminus B_R$, and $|\nabla\eta|\leq C$ where C is a positive constant, independent of R and n. Now test the identity $J_n'(v_n)=0$ against the function $\eta\bar{v}_n\in H^1_{A_n,V_n}$ to obtain

$$0 = J_n'(v_n)[\eta\overline{v_n}] = H_n + \int_{\varepsilon_n^{-1}(\Lambda_n - x_n)\setminus B_R} (2\sigma_n + g_n)$$

with $g_n(x)=2G(x_n+\varepsilon_n x,|v_n|^2)-g(x_n+\varepsilon_n x,|v_n|^2)|v_n|^2$ and

$$H_n = \mathrm{Re}\int_{B_R\setminus B_{R-1}} \nabla v_n \cdot \overline{\nabla(\eta v_n)} + \int_{B_R\setminus B_{R-1}} |A(x_n+\varepsilon_n x)|^2 \eta |v_n|^2 - \mathrm{Re}\frac{2}{i}\int_{B_R\setminus B_{R-1}} A(x_n+\varepsilon_n x)\cdot \nabla v_n \eta \overline{v_n}$$
$$+ \int_{B_R\setminus B_{R-1}} V(x_n+\varepsilon_n x)\eta |v_n|^2 - \int_{B_R\setminus B_{R-1}} g(x_n+\varepsilon_n x, |v_n|^2)\eta |v_n|^2.$$

From the local C^1 convergence of $\{v_n\}$ to v and the fact that $v\in H^1(\mathbb{R}^N,\mathbb{C})$, we deduce that there exists $R>0$ so large that $\lim_{n\to\infty}|H_n|\leqslant\delta$. Recalling that $g_n\leqslant 0$ because of the properties of g, one easily gets (48). But $V(x^*)\geqslant c-\delta$ so that $I_{V(x^*)}(v)\geqslant b^{c-\delta}$. We conclude that $b_n\geqslant(b^{c-\delta}+o(1))\varepsilon^3$. Equation (37) follows easily from (42) and (38).

Step III: We now introduce the well-known tool of the *center of mass* for an L^2 function, and apply it to our ϕ_n.

Let $u\in L^2(\mathbb{R}^3)$ be a given map. We define its center of mass $\beta(u)\in\mathbb{R}^3$ as

$$\beta(u) = \frac{\int_{\Lambda^+} x|u(x)|^2\,\mathrm{d}x}{\int_{R^2}|u(x)|^2\,\mathrm{d}x},$$

where Λ^+ is a fixed small neighborhood of $\overline{\Lambda}$. We may of course assume that $\delta_0<\mathrm{dist}(\partial\Lambda^+,\overline{\Lambda})$ where δ_0 is fixed in (27). We claim that

$$\beta(\phi_n(y)) \in \Lambda^+ \cap \left\{ x\in\mathbb{R}^3 \,\middle|\, V(x)\leqslant c-\frac{\delta}{2}\right\} \quad \text{for all } y\in B^{\varepsilon_n}. \tag{49}$$

Again, the proof of this fact is by contradiction. If (49) is false, then, passing to some subsequence, the existence of $y_n\in B^{\varepsilon_n}$ is assured, such that

$$\beta(\phi_n(y_n)) \notin \Lambda^+ \cap \left\{ x\in\mathbb{R}^3 \,\middle|\, V(x)\leqslant c-\frac{\delta}{2}\right\} \tag{50}$$

for all $n\in N$. If we set $u_n=\phi_n(y_n)$ and $v_n(x)=u_n(\varepsilon_n x)$, we have

$$\sup_{t>0} I_{c-\delta}(t|v_n|) \leqslant b^{c-\delta} + o(1). \tag{51}$$

This inequality is proved as follows: it is already known from (36) and (17) that $\{|v_n|\}$ is bounded in $H^1(\mathbb{R}^3,\mathbb{R})$. Moreover, it follows from

$$\int_{\mathbb{R}^3} (|D^{\varepsilon_n} v_n|^2 + V(\varepsilon_n x)|v_n|^2)\mathrm{d}x = \int_{\mathbb{R}^3} g(\varepsilon_n x, |v_n|^2)|v_n|^2 \leqslant \int_{\mathbb{R}^3} f(|v_n|^2)|v_n|^2$$

that

$$\inf_n \int_{\mathbb{R}^3} |v_n|^{p+1} = \sigma > 0.$$

By virtue of Lions' vanishing lemma (Ref. 25, Lemma I.1) we may find a sequence $\{B_n\}$ of balls of fixed radius (say 1) with

$$\inf_n \int_{B_n} |v_n|^2 \geqslant \sigma > 0.$$

For each n, select a number $t_n > 0$ such that $I_{c-\delta}(t_n|v_n|) = \sup_{t>0} I_{c-\delta}(t|v_n|)$. From the boundedness of $\{|v_n|\}$ in $H^1(\mathbb{R}^3,\mathbb{R})$, we get

$$Ct_n^2 - \int_{R^3} F(|t_n v_n|^2) \geqslant I_{c-\delta}(t_n|v_n|) \geqslant b^{c-\delta}.$$

Recalling assumption (H2), we have

$$t_n^{\vartheta-2}\int_{R^3} |v_n|^{\vartheta} \leqslant C$$

with $2 < \vartheta < 5$. Thus $\{t_n\}$ is bounded, and from (37) we have

$$\lim_{n\to\infty}\int_{\mathbb{R}^3\setminus(\varepsilon_n^{-1}\Lambda^+)} |t_n v_n|^2 = 0. \tag{52}$$

Finally, from (36) we have

$$b^{c-\delta} + o(1) \geqslant \varepsilon_n^{-3} J_{\varepsilon_n}(t_n u_n) \geqslant I_{c-\delta}(t_n|v_n|) - \frac{t_n^2}{2}\int_{\mathbb{R}^3\setminus(\varepsilon_n^{-1}\Lambda^+)} (c-\delta+o(1))|v_n|^2,$$

and (51) follows from (52).

Set now $w_n = t_n|v_n|$, with the same t_n as before. The function w_n belongs to the Nehari manifold of $I_{c-\delta}$ and (51) implies that w_n is a minimizing sequence of $I_{c-\delta}$ constrained on the Nehari manifold. A standard application of Ekeland's variational principle yields a (PS) sequence $\{\tilde{w}_n\}$ for $I_{c-\delta}$ such that $|w_n| - \tilde{w}_n \to 0$ in $H^1(\mathbb{R}^3,\mathbb{R})$. By concentration-compactness arguments, there exists a sequence $\{z_n\}$ of points in $\mathbb{R}^3$ such that $w_n(\cdot - z_n)$ converges in H^1 to some w which solves

$$\Delta w - (c-\delta)w + f(|w|^2)w = 0 \quad \text{in } \mathbb{R}^3.$$

Denote $\bar{y}_n = \varepsilon_n z_n$. From (37), we can assume that, up to a subsequence, $\bar{y}_n \to \bar{y}$ in $\overline{\Lambda}$. Since

$$b^{c-\delta} \geqslant \lim_{n\to\infty} \varepsilon_n^{-3} J_{\varepsilon_n}(t_n u_n) \geqslant \lim_{n\to\infty} I(t_n|v_n|) = I_{V(\bar{y})}(\omega),$$

we have $b^{c-\delta} \geqslant b^{V(\bar{y})}$, so that $V(\bar{y}) \leqslant c-\delta$. But $\beta(u_n) \to \bar{y} \in \overline{\Lambda}$ and (50) implies $V(\bar{y}) > c - \delta/2$. This contradiction proves (49).

Step IV: We are going to find a contradiction that proves (35), which will complete the proof. Recall the validity of (49). Let $\varphi_n(y) = \pi(\beta(\phi_n(y)))$ where π: $\Lambda^+ \to \Lambda$ is a continuous mapping that equals the identity on Λ and Λ^+ is a fixed small neighborhood of Λ fixed in Step III. Now, $\phi_n(y) = w_{\varepsilon_n}^{w}$ for each $y \in B_0^{\varepsilon_n}$, and $w_{\varepsilon_n}^{y}$ is radially symmetric with respect to the point y. Therefore ϕ_n acts as the identity on $B_0^{\varepsilon_n}$. As such, map ϕ_n is admissible in the class of functions that defines the level c. Assumption (V2) implies now that $c \leqslant \sup_{y\in B_0^{\varepsilon_n}} V(\varphi_n(y))$ for all $n \geqslant 1$. If n is large enough, this contradicts (49), and we have proved that (35) is true. □

Proposition 5.3: For each ε sufficiently small, the number γ_ε defined by (29) is a critical value for the functional J_ε. As a consequence, there exists a solution $u_\varepsilon \in H_{A,V}^{\varepsilon}$ to Eq. (23) such that $J_\varepsilon(u_\varepsilon) = \gamma_\varepsilon$. Furthermore $u_\varepsilon \in C_{\mathrm{loc}}^{2,\alpha}(\mathbb{R}^3)$, with $\alpha \in (0,1)$.

Proof: We already know from Lemma 4.1 that J_ε satisfies the (PS) condition, provided ε is small enough. Moreover, the last lemma implies that $\varepsilon^{-3}\gamma_\varepsilon \geqslant b^{c-\delta} + o(1)$ for all ε small. If $\phi \in \Gamma_\varepsilon$, then $\phi(y) = w_\varepsilon^{y}$ whenever $y \in B_0^{\varepsilon}$. This entails that

$$\sup_{y\in B_0^\varepsilon} \varepsilon^{-3} J_\varepsilon(\phi(y)) \leqslant b^{c-\delta} + o(1).$$

The proof is completed by a standard deformation argument. By standard regularity results, $u_\varepsilon \in C^{2,\alpha}_{\mathrm{loc}}(\mathbb{R}^3)$. □

VI. PROOF OF THE MAIN THEOREM

First we derive the following proposition in which the asymptotic behavior of $\max_{\partial\Lambda}|u_\varepsilon|$ is described.

Proposition 6.1: Let

$$m_\varepsilon = \max_{x\in\partial\Lambda}|u_\varepsilon(x)|,$$

then

$$\lim_{\varepsilon\to 0} m_\varepsilon = 0. \tag{53}$$

Proof: We split again the proof.

Step I: We begin to establish the following fact: if $\varepsilon_n \to 0$ and $x_n \in \overline{\Lambda}$ are such that $|u_{\varepsilon_n}(x_n)| \geqslant b > 0$, then

$$\limsup_{n\to\infty} V(x_n) \leqslant c.$$

By contradiction, we assume, up to a subsequence, that $x_n \to x^* \in \overline{\Lambda}$ and $V(x^*) > c$. Set $v_n(x) = u_{\epsilon_n}(x_n + \epsilon_n x)$, we have that v_n satisfies the equation

$$- \Delta v_n - \frac{2}{i} A(x_n + \varepsilon_n x) \cdot \nabla v_n + |A(x_n + \epsilon_n x)|^2 v_n - \frac{\varepsilon_n}{i} \mathrm{div}\, A(x_n + \varepsilon_n x) v_n$$
$$+ V(x_n + \varepsilon_n x) v_n = g(x_n + \varepsilon_n x, |v_n|^2) v_n \quad \text{in } \mathbb{R}^3 \tag{54}$$

By reasoning as in Step II of Lemma 2, $\{v_n\}$ converges in $C^2_{\mathrm{loc}}(\mathbb{R}^3)$ to some v. Let χ be the weak * limit in $L^\infty(\mathbb{R}^3)$ of the sequence $\{\chi_\Lambda(x_n + \varepsilon_n \cdot)\}$, $v \in C^2_{\mathrm{loc}}(\mathbb{R}^3)$ solves the equation in each compact set

$$- \Delta v - \frac{2}{i} A(x^*) \cdot \nabla v + |A(x^*)|^2 v + V(x^*) v = \bar{g}(x, |v|^2) v \quad \text{in } \mathbb{R}^3, \tag{55}$$

where $\bar{g}(x,s) = \chi(x) f(s) + (1-\chi(x))\tilde{f}(s)$ and $0 \leqslant \chi \leqslant 1$. Since $v \in H^1(\mathbb{R}^3, \mathbb{C})$, we infer v solves (46) in weak sense. Setting $\tilde{v}(x) = e^{-iA(x^*)\cdot x} v(x)$, we see that $\tilde{v}$ weakly solves

$$- \Delta \tilde{v} + V(x^*) \tilde{v} = \bar{g}(x, |\tilde{v}|^2) \tilde{v} \quad \text{in } \mathbb{R}^3 \tag{56}$$

Let $\bar{J}$: $H^1(\mathbb{R}, \mathbb{C}) \to R$ be the functional defined by

$$\bar{J}(u) = \frac{1}{2} \int_{\mathbb{R}^3} |\nabla u|^2 + V(x^*)|u|^2 - \int_{\mathbb{R}^3} \bar{G}(x, |u|^2),$$

where $\bar{G}(x,s) = \int_0^s \bar{g}(x,t)\mathrm{d}t$, we observe that $\tilde{v}$ is a critical point of $\bar{J}$. Following Step II of Lemma 5.2, one can prove that

$$\bar{J}(\tilde{v}) \leqslant \liminf_{n\to\infty} J_{\varepsilon_n}(v_n). \tag{57}$$

By (57) and taking into account (30) we deduce that $b^c \geqslant \bar{J}(\tilde{v})$. Since $h(s) \geqslant \tilde{h}(s)$ for all s we derive

$$b^c \geqslant \bar{J}(\tilde{v}) = \max_{\tau\geqslant 0} \bar{J}(\tau v) \geqslant \max_{\tau\geqslant 0} I_{V(x^*)}(tv) \geqslant b^{V(x^*)},$$

where

$$I_{V(x^*)}(u) = \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u|^2 + V(x^*)|u|^2 - \int_{\mathbb{R}^3} F(|u|^2).$$

It follows that $V(x^*) \leqslant c$, which contradicts the fact that $V(x^*) > c$.

Step II: Now we pass to prove (53). By contradiction, we assume, up to a subsequence, that there exists a sequence $x_n \in \partial\Lambda$ such that $x_n \to \bar{x} \in \overline{\partial\Lambda}$ and

$$|u_{\varepsilon_n}(x_n)| \geqslant \gamma > 0. \tag{58}$$

It follows that $V(\bar{x}) \leqslant c$. We claim that $V(\bar{x}) > c - \delta$. By contradiction, we suppose that $V(\bar{x}) = c - \delta$.

Arguing as before, we can consider the scaled sequence $v_n(x) = u_{\varepsilon_n}(x_n + \varepsilon_n x)$, and we can deduce that v_n solves (45) and it converges to some $v \in H^1(\mathbb{R}^3, \mathbb{C})$ in $C^2_{\mathrm{loc}}(\mathbb{R}^3)$, up to subsequences and $v \neq 0$. Moreover v weakly solves the equation

$$-\Delta v - \frac{2}{i}A(\bar{x}) \cdot \nabla v + |A(\bar{x})|^2 v + V(\bar{x})v = \bar{g}(x, |v|^2)v \quad \text{in } \mathbb{R}^3, \tag{59}$$

where $\bar{g}(x,s) = \chi(x)f(s) + (1-\chi(x))\tilde{f}(s)$.

Setting $\tilde{v}(x) = e^{-iA(\bar{x})\cdot x}v(x)$, we see that $\tilde{v}$ weakly solves

$$-\Delta\tilde{v} + V(\bar{x})\tilde{v} = \bar{g}(x, |\tilde{v}|^2)\tilde{v} \quad \text{in } \mathbb{R}^3 \tag{60}$$

and thus $\tilde{v}$ is a critical point of the functional $\bar{I}$: $H^1(\mathbb{R}^3, \mathbb{C}) \to \mathbb{R}$ defined by

$$\bar{I}(u) = \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u|^2 + V(\bar{x})|u|^2 - \int_{\mathbb{R}^3} \bar{G}(x, |u|^2),$$

where $\bar{G}(x,s) = \int_0^s \bar{g}(x,t)\mathrm{d}t$.

Now for any $n \in N$ we consider the positive measure $\mu_n(\Omega) = \int_\Omega |\nabla |v_n||^2 + V(x_n + \varepsilon_n x)|v_n|$. We have that the sequence $\{\mu_n(\mathbb{R}^3)\}_n$ is bounded and, up to subsequences, μ_n tends to some $\tilde{c}$.

Therefore there exists a subsequence of $\{\mu_n\}_n$ (without relabelling) for which one of the three possibilities of Lions' concentration-compactness lemma (see Ref. 25) holds. First we notice that vanishing cannot occur, as $|v(0)| > 0$.

If we have tightness, we derive that there exists z_n with the following property: for any $\gamma > 0$ there exists $\rho > 0$ such that

$$\int_{B_\rho(z_n)} |\nabla |v_n||^2 + V(x_n + \varepsilon_n x)|v_n| \geqslant \tilde{c} - \gamma.$$

If $\varepsilon_n z_n$ tends to some point $y \in \mathbb{R}^3$, then we derive that $|v_n|$ tends to $|v|$ strongly in $H^1(\mathbb{R}^3, \mathbb{R})$ and thus $|v_n|$ tends to $|v|$ strongly in $L^q(\mathbb{R}^3, \mathbb{R})$ with $q < 5$.

Testing Eq. (45) we get that

$$\int_{\mathbb{R}^3} \left| \left(\frac{\nabla}{i} - A_n(x) \right) v_n \right|^2 + V_n(x)|v_n|^2 = \int_{\mathbb{R}^3} \bar{g}(x, |v_n|^2)|v_n|^2$$

and, since $\int \bar{g}(x, |v_n|^2)|v_n|^2 \to \int \bar{g}(x, |v|^2)|v|^2$ and v solves (60) we deduce

$$\int_{R^3} \left| \left(\frac{\nabla}{i} - A_n(x) \right) v_n \right|^2 + V_n(x)|v_n|^2 \to \int_{R^3} \left| \left(\frac{\nabla}{i} - A(\bar{x}) \right) v \right|^2 + V(\bar{x})|v|^2 \tag{61}$$

and so

$$b^c \geqslant \lim_n \varepsilon_n^{-3} J_{\varepsilon_n}(u_n) = \bar{I}(\tilde{v}).$$

After a rotation, by Lemma 5.1, $\tilde{v}$ solves

$$-\Delta \tilde{v} + V(\bar{x})\tilde{v} = f(|\tilde{v}|^2)\tilde{v}$$

and since we have assumed $V(\bar{x}) = c - \delta$, we get $\bar{I}(\tilde{v}) = b^{c-\delta}$. This is a contradiction to (35).

Conversely, if $|\varepsilon_n z_n| \to \infty$, we can conclude that

$$b^c \geqslant \lim_n \varepsilon_n^{-3} J_{\varepsilon_n}(u_n) = b^{c-\delta} + b^{V_0}$$

which, by the continuity of $a \mapsto b^a$, is not possible if δ is chosen sufficiently small. In a similar way we can infer that dichotomy cannot occur.

Therefore $V(\bar{x}) > c - \delta$ and $|v_n| \to |v|$ strongly in $H^1(\mathbb{R}^3, \mathbb{R})$ as $n \to +\infty$.

Step III: Observe that we can assume that δ was fixed so that $\bar{x}$ lies in a region where $\partial\Lambda$ is smooth and $\partial_\tau V(\bar{x}) \neq 0$.

Arguing as in Ref. 14 we can assume $\bar{x} = 0$ and the domain Λ can be described as

$$\Lambda \cap B(0,2\rho) = \{(x,x') \in B(0,2\rho) | x' \in \mathbb{R}^2,\ x_3 < \psi(x')\},$$

where ψ is a smooth function such that $\psi(0) = 0$ and $\nabla\psi(0) = 0$. So we have that in $B(0, \rho/\varepsilon_n) v_n$ satisfies

$$\begin{aligned} &-\Delta v_n - \frac{2}{i} A(x_n + \varepsilon_n x) \cdot \nabla v_n + |A(x_n + \varepsilon_n x)|^2 v_n - \frac{\varepsilon_n}{i} \operatorname{div} A(x_n + \varepsilon_n x) v_n + V(x_n + \varepsilon_n x) v_n \\ &= \chi_{\{z_3 < \varepsilon^{-1}\psi(\varepsilon z')\}} f(|v_n|^2) v_n + \chi_{\{z_3 > \varepsilon^{-1}\psi(\varepsilon z')\}} \tilde{f}(|v_n|^2) v_n. \end{aligned} \tag{62}$$

Since $|v_n|$ converges to $|v|$ strongly in $H^1(\mathbb{R}^3, \mathbb{R})$, arguing as in Ref. 21 or in Ref. 31 we derive that $|v_n(z)| \leqslant Ce^{-\beta|z|}$ for some constants C, β independent of n. Recalling that each v_n is complex valued, it is not so easy to prove a similar decaying behavior for the gradients ∇v_n, too. Hence we need to modify the proof in Ref. 14. The main tool is a kind of variational identity inspired to the celebrated Pucci–Serrin identity in Ref. 28 (see also Ref. 12). Since all the details for deriving this identity for complex-valued solutions to the Schrödinger equation with magnetic field can be found in Ref. 31, we will be rather sketchy. Fix the index $n \geqslant 1$, and choose a sequence $\{\psi_h\}_{h \in \mathbb{N}}$ of functions from $C_0^\infty(B(0, \rho/\varepsilon_n))$ such that their supports converge to $B(0, \rho/\varepsilon_n)$ as $h \to +\infty$. Now multiply equation (62) by $\psi_h(\partial v_n / \partial x_k)$ $(k = 1, 2)$ and integrate by parts. By reasoning as in Ref. 31 and exploiting (8) and (9), we can show that it is possible to take first the limit as $h \to \infty$ and then the limit $n \to \infty$, finally deducing that

$$\begin{aligned} &\int_{\mathbb{R}^3} \frac{\partial A}{\partial x_k}(0) \cdot A(0)|v|^2 \, dx - \mathrm{Re} \frac{1}{i} \int_{\mathbb{R}^3} \nabla v \cdot \frac{\partial A}{\partial x_k}(0) \bar{v} \, dx + \frac{\partial V}{\partial x_k}(0) \int_{\mathbb{R}^3} \frac{|v|^2}{2} dx \\ &= \int_{\mathbb{R}^2} \left[F(|v|^2) - \tilde{F}(|v|^2) \right] z' \cdot \nabla \frac{\partial \psi}{\partial x_k}(0) dz'. \end{aligned}$$

If we define U_0 by $v(x) = e^{iA(0)\cdot x} U_0(x)$, then $U_0 \in H^1(\mathbb{R}^3, \mathbb{C})$ satisfies the identity

$$-\Delta U_0 + V(0)U_0 = \chi_{\{z_3<\psi(0)\}} f(|U_0|^2)U_0 + \chi_{\{z_3>\psi(0)\}} \tilde{f}(|U_0|^2)U_0.$$

Hence by Lemma 5.1 (after a suitable rotation) and an elementary calculation we conclude that

$$\frac{\partial A}{\partial x_k}(0) \cdot \int_{\mathbb{R}^3} \mathrm{Re}(i\bar{U}_0 \nabla U_0)\mathrm{d}x + \frac{\partial V}{\partial x_k}(0)\int_{\mathbb{R}^3} \frac{|U_0|^2}{2}\mathrm{d}x = 0.$$

But by the uniqueness of critical points for the functional I_a [see assumption (f2) and the arguments in Ref. 31], $\mathrm{Re}(i\bar{U}_0 \nabla U_0)=0$ a.e. in $\mathbb{R}^3$. This immediately implies that $(\partial V/\partial x_k)(0)=0$ for $k=1,2$ and so $\partial_\tau V(\bar{x})=0$. This contradiction complete the proof. □

Finally we prove the main result. For the reader's convenience, we repeat its statement below.

Main Theorem: *Under assumptions* (f1) and (f2), (A1), (V1–V3), *there is a number* $\varepsilon_0>0$ *such that for all* $\varepsilon<\varepsilon_0$, *there exists a solution* $u_\varepsilon \in H^\varepsilon_{A,V}$ *of* Eq. (6). *Furthermore* $u_\varepsilon \in C^{2,\alpha}_{\mathrm{loc}}(\mathbb{R}^3)$ *with* $\alpha\in(0,1)$.

Proof: By Proposition 6.1, for all ε small enough,

$$|u_\varepsilon(x)| < \sqrt{a} \quad \text{for all } x \in \partial\Lambda.$$

The function u_ε satisfies the equation

$$\left(\frac{\varepsilon}{i}\nabla - A\right)^2 u_\varepsilon + V u_\varepsilon = g(x,|u_\varepsilon|^2)u_\varepsilon \quad \text{in } \mathbb{R}^3. \tag{63}$$

Therefore we can test (63) against $(|u_\varepsilon|-\sqrt{a})_+$, and recalling Kato's inequality we find

$$\int_{\mathbb{R}^3\setminus\Lambda} \varepsilon^2|\nabla(|u_\varepsilon| - \sqrt{a})_+|^2 + c(x)(|u_\varepsilon| - \sqrt{a})_+^2 + c(x)\sqrt{a}(|u_\varepsilon| - \sqrt{a})_+ \leqslant 0, \tag{64}$$

where

$$c(x) = V(x) - g(x,|u_\varepsilon(x)|^2).$$

By definition of g, we have $c>0$ in $\mathbb{R}^3\setminus\Lambda$. Hence all terms in (64) are necessarily zero, and in particular

$$|u_\varepsilon(x)| \leqslant \sqrt{a} \quad \text{for all } x \in \mathbb{R}^3 \setminus \Lambda.$$

This, of course, implies that u_ε is a solution of (6). □

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